

Theta Functions  
FAMAT State Convention 2003

**The abbreviation NOTA denotes  
"None of These Answers."**

1.

$x$	$f(x)$	$g(x)$
1	1	2
2	3	1
3	2	3

Some values of functions  $f$  and  $g$  are shown in the table above. If both functions are defined for all values of  $x$  such that  $0 < x < 4$  and  $h(x) = f(g(x))$  then  $h(1) =$

- A. 1                      B. 2  
C. 3                      D. not given  
E. NOTA

2. If  $f(x) = \sqrt{2x+1}$  then what is the solution set of all possible values of  $x$  for which  $f(x) \geq 2$  ?

- A.  $x > 5$       B.  $x \geq \sqrt{5}$   
C.  $x \geq 1.5$     D.  $x \geq -0.5$     E. NOTA

3. If  $f(x) = \sqrt{x}$  and  $a < 0$  and  $i = \sqrt{-1}$  then  $f(a) \cdot f(a) =$

- A.  $a$                       B.  $-a$   
C.  $-i\sqrt{a}$               D.  $-ai$               E. NOTA

4. Which is/are (an) even function(s) ?

- i)  $y = 3x^2$   
ii)  $y = (x+1)^2$   
iii)  $y = x^2 + 1$   
iv)  $y = x^3 - x^2$

- A. i, iii only              B. i, ii, iii only  
C. ii, iv only             D. none of them  
E. NOTA

5. Let  $f(x)$  be an even function and  $g(x)$  be an odd function, both defined for all real numbers  $x$ . If  $f(1) = g(1) \neq 0$  then which of the following is NOT true?

- A.  $f(-1) = g(1)$   
B.  $f(-1) = g(-1)$   
C.  $f(-1) = -g(-1)$   
D.  $f(1) = -g(-1)$   
E. NOTA

6. Square RSTU has vertices  $(0, 0)$ ,  $(5, 0)$ ,  $(0, 5)$  and  $(5, 5)$ . Square ABCD has two sides with slope 1, has area 25 sq. units and has diagonal  $\overline{AC}$  on the positive  $x$ -axis. If  $f(x)$  is defined as the area of the intersection of the interiors of ABCD and RSTU for  $x > 0$ , when A lies on point  $A(x, 0)$ , and C lies on the  $(d, 0)$  for  $d > x$  then what is the least integer value of  $x$  for which  $f(x) < 2$  ?

- A. 2                      B. 3  
C. 4                      D. 24                      E. NOTA

7. In a parabola with equation  $y = f(x)$  the latus rectum has endpoints  $(0, 0)$  and  $(8, 0)$ . What is the value of  $f(2)$  if  $f(2) > 0$  ?

- A. 1.5                      B. 1.75  
C. 2                        D. 2.5                      E. NOTA

8. If  $f(x) = \frac{1}{1 + \frac{1}{x}}$  then give the value

of  $f\left(f\left(\frac{3}{4}\right)\right)$ .

- A.  $\frac{7}{11}$                       B.  $\frac{3}{10}$   
C.  $\frac{7}{4}$                         D.  $\frac{7}{10}$                       E. NOTA



Theta Functions  
FAMAT State Convention 2003

18. For  $f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^n$  find the value of  $f(4) - f(5)$ .

- A. 1                      B.  $\frac{19}{20}$   
C.  $\frac{-39}{4}$                   D.  $\frac{1}{12}$                   E. NOTA

19. The function  $f$  is defined by

$$f(x) = \begin{cases} x+1 & \text{for } x > 1 \\ -2x^2 & \text{for } x \leq 1 \end{cases}$$

has a range of  $y > a$  or  $y \leq b$ .  
Give the value of the difference (a-b).

- A. 4                      B. 2  
C. 0                      D. -2                      E. NOTA

20. Let  $f(x) = \sqrt{9 - \frac{4}{9}x^2}$  for  $|x| \leq 4.5$ . A

rectangle ABCD has base  $\overline{AB}$  on the x-axis and C on the graph of  $f$ . The midpoint of  $\overline{AB}$  is (0,0). The area of ABCD is  $6\sqrt{5}$ . What is the length of the diagonal  $\overline{AC}$  ?

- A.  $\sqrt{14}$                   B.  $2\sqrt{14}$   
C.  $\sqrt{41}$                   D.  $12\sqrt{5}$                   E. NOTA

21. One root of  $f(x) = x^3 - 6x^2 - kx + 5$ , for some real constant  $k$ , is 5. If the other two roots are  $U$  and  $V$  then find the value of  $U^2 + V^2$ .

- A. 3                      B. 2  
C. 0                      D. -1                      E. NOTA

22. The function  $f(x)$  gives the height of a tree, which increases  $\frac{1}{2}\%$  in height each year. If  $f(1) = 1$  ft ( $x=1$  indicates that one year has passed) then what is the value of  $f(10)$  to the nearest hundredth of a foot?

- A. 10.23                  B. 9.18  
C. 1.05                  D. 1.04                  E. NOTA

23. Let  $f$  be a linear function with slope  $m$  and y-intercept zero. Let  $g(x) = -2(x-2)$  have x-intercept at point A. For the x-values for which the graphs of  $f$  and  $g$  intersect (at point C) with point B on the origin, how many values of  $m$  will make the area of triangle ABC equal to 9 ?

- A. 0                      B. 1  
C. 2                      D. 3                      E. NOTA

24. In a circle, chord  $\overline{AB}$  is parallel to diameter  $\overline{CD}$  which has length 12. Let  $f(x)$  be the distance from  $\overline{AB}$  to  $\overline{CD}$  when  $\overline{AB}$  has length  $x$ . The range of  $f$  is real numbers  $y$  such that  $0 \leq y \leq 6$ . If  $f(k) = 3$  then  $k = ?$

- A.  $3\sqrt{3}$                   B.  $6\sqrt{3}$   
C.  $3\sqrt{15}$                   D.  $6\sqrt{15}$                   E. NOTA

25. Let  $f(x)$  be a polynomial function of degree 3 with roots  $1+i$  and  $\frac{1}{3}$ . If the y-intercept of  $f$  is 6 then what is the value of  $f(1)$ ?

- A. 10                      B. 3  
C. 2                      D. -6                      E. NOTA

Theta Functions  
FAMAT State Convention 2003

26. Let  $f(x) = C(x, x-2)$  where  $C(a, b)$  denotes the number of combinations possible of  $a$  objects taken  $b$  at a time. If the value of  $f(10) = f(k) - 75$  then find the value of  $k$ .

- A. 10            B. 12  
C. 14            D. 16            E. NOTA

27.  $f(g(x)) = g(f(x)) = x$  and  $f(2) = g(3)$  and the graph of  $g$  contains the point  $(4, 2)$ . If  $f$  and  $g$  are defined for all Reals, then which must be true?

- A.  $f(4) = 3$             B.  $g(2) = 4$   
C.  $f(4) = 2$             D.  $f(3) = 4$   
E. NOTA

28. A parabola has an axis of symmetry which is vertical, and the focus  $F$  is 8 units from the vertex  $V$ . How wide is the graph of the parabola (horizontal width) at a point which is one unit from the vertex and 7 units from the focus?

- A. 32            B. 16  
C.  $8\sqrt{2}$             D.  $4\sqrt{2}$             E. NOTA

29. For  $x \geq 1$ ,  $x$  fair coins are tossed simultaneously. It is known that at least one coin lands heads side up. Let  $f(x)$  be the probability that all of the coins land heads side up. If  $f(k) = \frac{1}{15}$  then what is the value of  $k$ ?

- A. 3            B. 4  
C. 5            D. 16            E. NOTA

30. A right triangle has integral length sides. If one side has length 15 then give the sum of all possible hypotenuses for the triangle.

- A. 221            B. 209  
C. 96            D. 32            E. NOTA

Solutions:

1.  $h(1) = f(g(1)) = f(2) = 3$ . **Choice C.**

# Theta Functions

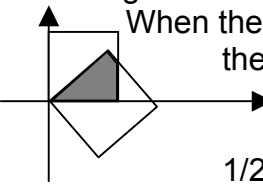
## FAMAT State Convention 2003

2.  $\sqrt{2x+1} \geq 2$ ;  $2x+1 \geq 4$ ;  $x \geq 1.5$ . **Choice C.**

3. Consider  $a = -2$ :  $\sqrt{-2} \cdot \sqrt{-2} = i^2 \sqrt{4} = -2$  and since  $a = -2$  then this equals  $a$ . Likewise, any negative value of  $a$  yields a negative answer, equal to  $a$ . **Choice A.**

4.  $f(-a) = f(a)$  for all values of  $a$  in the domain. Functions i and iii: **Choice A.**

5. Since  $f$  is even,  $f(1) = f(-1)$  and since  $g$  is odd,  $g(1) = -g(-1)$ . So at  $x=1$ ,  $f$  and  $g$  are both equal to  $a$ , and at  $x=-1$ ,  $f = a$  and  $g = -a$ . So **choice B** is correct.

6.  When the vertex  $C$  meets the right side of square  $RSTU$  the area enclosed is  $\frac{1}{2}(2.5)(2.5) = 3.125$ . So

for the area to be smaller,  $x$  must be greater than  $2.5$ . Let  $x=3$ . We get area  $2$ . So for area to be less than  $2$  we must use  $x=4$ .

**Choice C.**

7. The roots of  $f$  are  $(0,0)$  and  $(8,0)$ . Also the length of the latus rectum is  $8$  so the equation of the curve is  $y = \frac{-1}{8}(x-0)(x-8)$

and  $f(2) = 1.5$ . **Choice A.**

8.  $f(3/4) = 3/7$  and  $f(3/7) = 3/10$ . **Choice B.**

9. Using the change of base rule, we have

$$\log_2 3 = \frac{\log 3}{\log 2} \text{ and } f = \frac{\log x}{\log 2}; g = \frac{\log x}{\log 3}$$

$$\text{Dividing } f \text{ by } g \text{ gives } \frac{\log x / \log 2}{\log x / \log 3} = \frac{\log 3}{\log 2}$$

**Choice D.**

10. The area of the annulus is  $\pi(x+1)^2 - \pi x^2$ . Setting this equal to  $12$  gives approximately  $x=1.4$ . **Choice A.**

11. The graphs meet where  $0.5x+2 = x$  for the positive side of the absolute value graph, and where  $0.5x+2 = -x$  for the negative side of the abs. value graph. The points of intersection are  $(4/3, 4/3)$  and  $(-4/3, 4/3)$ . Using the distance formula gives  $\frac{\sqrt{320}}{3} = \frac{8\sqrt{5}}{3}$ . **Choice B.**

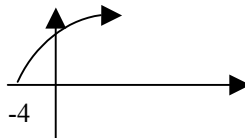
12. Using the formula for the  $n$ th term of an arithmetic sequence, we get  $g(100) = 1 + (100-1)4$ ;  $f(100) = 4 + (100-1)3$ .

the difference of these two terms is  $96$ .

**Choice A.**

13. Since one root is  $1 + \sqrt{3}$  then its conjugate is also a root. So  $(1 + \sqrt{3}) + (1 - \sqrt{3}) + c = 3$  gives the last root to be  $1$ . The product is then  $(1 - \sqrt{3})(1 - \sqrt{3})(1) = -2$ . **Choice B.**

14. Since the graph, for  $p=0$  looks like the picture below, raising the graph will make it have no real roots.  $p > 0$ . **Choice A.**



15.  $4i^5 + (1+i)^{10} = 4i^5 + (2i)^5$  since the parentheses to the power  $2$  reduces to  $2i$ . So we have  $4i + 32i = 36i$ . **Choice A.**

16. The domain of  $f$  is positive reals, so **choice D** will work.

17.  $2(2)(k) - 5 = 7$  solves to  $k=3$ . **Choice B.**

18. The function  $f$  gives an infinite series, the sum of which is given by the formula  $\frac{a_1}{1-r}$

$$\text{so } f(4) = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3} \text{ and } f(5) = \frac{1}{4}$$

difference is  $1/12$ , **choice D.**

19. For  $x$  greater than  $1$ , the graph is a line from the point  $(1,2)$  which is not on the graph. For  $x$  less than  $1$  the graph is a parabola, opening downward with vertex at the origin. The parabola has range from  $0$  (inclusive) to negative infinity. The line has range  $2$  (not included) to infinity. The total range is  $y > 2$  or  $x \leq 0$ . So  $a-b = 2$ . **Choice B.**

20. This is the top part of an ellipse with the rectangle having area  $(2x)$  times the height of graph  $f$ . So

$$2x\sqrt{9 - \frac{4}{9}x^2} = 6\sqrt{5} \text{ which by inspection is}$$

easily seen to have  $2x=6$  and  $x=3$  (check, of course). So the diagonal would be

Solutions: (continued)

20. (continued) from  $(-3, 0)$  to  $(3, \sqrt{5})$  and by

# Theta Functions

## FAMAT State Convention 2003

using the distance formula we get the length  $\sqrt{41}$ . **Choice C.**

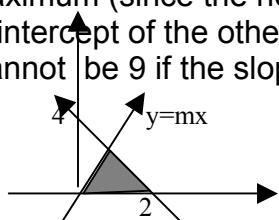
21.  $5 + U + V = 6$ , so  $U+V=1$  from the sum of the roots theorem  $(-B/A)$ . The product of the roots is the  $-constant/A$  which gives  $5UV = -5$ , and  $UV = -1$ . Squaring the first equation gives

$(U + V)^2 = U^2 + 2UV + V^2 = 1$  and since  $UV = -1$ , we have the sum of the squares of  $U$  and  $V$  are  $1+2(1) = 3$  (the last roots are complex). **Choice A.**

22. Year two would be 100% plus 0.5% which is 1.005 times the original height. So after year 10, if we use year 1 as our first term, we need nine more years:  $1(1.005)^9$  which gives 1.05 to the nearest hundredth.

The function  $f(x) = (1.005)^{x-1}$  gives  $f(1)=1$  and  $f(10)=1.0459$ . **Choice C.**

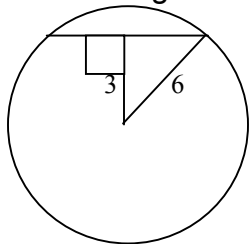
23. If the slope is positive such that the lines meet in quadrant I, then the area is less than 9:  $1/2 * base * height = 1/2 * 2 * 4$  is the maximum (since the height is greatest at the y-intercept of the other line, 4). So the area cannot be 9 if the slope is positive. If the



slope is negative and the lines are not parallel then they meet in quadrant II or IV. Two

possible triangles. **Choice C.**

- 24.



Using the Pythagorean Th gives the side of the triangle to be  $3\sqrt{3}$  and the

length of the chord is double that, **choice B.**

25. Using  $1+i$ , and  $1-i$  to get a quadratic with sum of roots  $-B/A$ , and product  $C/A$  gives  $(x^2 - 2x + 2)$  so the cubic must be

$f(x) = a(x^2 - 2x + 2)(x - \frac{1}{3})$  and since

25. Continued.

y-intercept is 6, we let  $x=0$  and get  $(-2/3)a$

equals 6 and  $a = -9$ . So now we let  $x=1$  and get  $(1-2+2)(1-1/3)(-9) = -6$ . **Choice D.**

26.  $C(10,8) = \frac{10!}{8!2!} = 45$ . So  $C(k, k-2) = 45$  and

$\frac{k!}{(k-2)!(2!)} - 75 = 45$  which reduces to

$\frac{k(k-1)}{2} = 120$  which solves to  $k=16$ . **Choice D**

27. Since  $f$  and  $g$  are inverses if  $f(2)=K$  then  $g(K)=2$ . So we have the chart below by the

x	f	g
2	4	?
3	?	4
4	3	2

information in the problem and the only statement which must be true is **choice A.**

28. Let the vertex be at the origin and so the equation may be  $y = \frac{1}{32}x^2$  with the

coefficient determined by  $\frac{1}{4a} = 8$ . Let

$y=1$  and we get the  $x$  value is  $\pm 4\sqrt{2}$  which gives a distance across  $8\sqrt{2}$ , **choice C.**

29. If there are two coins then the choices are HH, TT, TH and HT. Since we know one coin is heads, then TT is out. In general

there will be  $2^n$  choices for  $n$  coins, less one since TT is out. So if the probability is one chance (only one possible HH or HHH, etc) out of 15 then  $n$  must have been 4.

**Choice B.**

30. We have 3(3-4-5) and 3(5-12-13) and 1(8-15-17) 5(3-4-5) and one more. The formulae  $2uv$ ,  $u^2 - v^2$ ,  $u^2 + v^2$  give triples, so

let  $2uv=15$ . Not integers. Let  $u^2 - v^2 = 15$  which gives  $u-v=3$ ,  $u+v=5$  or  $u-v=1$ ,  $u+v=15$ . Solving both systems we get one repeat answer and also the triple 15, 112, 113. The sum of the hypotenuses =  $25 + 15 + 39 + 17 + 113 = 209$ . **Choice B.** By the way, there are few

integer values of  $u$  and  $v$  that produce a hypotenuse of 15, so we can do that part by trial and error to eliminate possibilities.