Limits & Derivatives (Calculus) — Solutions

FAMAT State Convention 2003

1. $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}} = \left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\sqrt{x}\right) = 1\times 0 = 0.$

2. $\displaystyle\left(\lim_{x\to n^-}\lfloor x\rfloor\right)\left(\lim_{x\to n^+}\lfloor x\rfloor\right) = (n-1)n = 2\sum_{i=1}^{(n-1)} i = \sum_{i=1}^{(n-1)}(2i) = 2+4+\cdots+2(n-1)$, none of which is a choice.

3. $f(1) = 7$, and $D_x[7] = 0$.

4. Since $|(4x+5)-45| < \epsilon \Leftrightarrow |4x-40| < \epsilon \Leftrightarrow |x-10| < \dfrac{\epsilon}{4}$, the largest value one could use is $\delta = \dfrac{\epsilon}{4}$.

5. Since $-M|f(x)| \le f(x)g(x) \le M|f(x)|$, by the squeezing theorem the limit must be 0.

6. Since $\dfrac{d}{du}\arctan u = \dfrac{1}{1+u^2}$, by letting $u = \dfrac{1}{x+1}$ and applying the chain rule we get the following derivative: $\displaystyle\frac{d}{dx}\arctan\left(\frac{1}{x+1}\right) = \frac{1}{1+\left(\frac{1}{x+1}\right)^2}\times\frac{-1}{(x+1)^2} = \frac{-1}{(x+1)^2+1} = \frac{-1}{x^2+2x+2}.$

7. $\displaystyle\lim_{x\to-\infty}\frac{x+2}{\sqrt{x^2+4}} = \lim_{x\to-\infty}\frac{1+\frac{2}{x}}{-\sqrt{1+\frac{4}{x^2}}} = \frac{1}{-1} = -1.$

8. $\displaystyle\lim_{h\to 0}\frac{f(x+ah)-f(x+bh)}{h} = \lim_{h\to 0}\left[a\times\frac{f(x+ah)-f(x)}{ah} - b\times\frac{f(x+bh)-f(x)}{bh}\right] = af'(x)-bf'(x) = (a-b)f'(x).$

9. Since $\ln\sqrt{1-\sin^2 x} = \ln\cos x$, the derivative is $\dfrac{-\sin x}{\cos x} = -\tan x$.

10. $\displaystyle\frac{dx}{dt} = \frac{(5t+2)(-2)-(3-2t)(5)}{(5t+2)^2} = \frac{-19}{(5t+2)^2}$, and $\displaystyle\frac{d^2x}{dt^2} = \frac{190}{(5t+2)^3}$, so at $t=2$ the acceleration is $\displaystyle\frac{190}{(5\times 2+2)^3} = \frac{95}{864}$.

11. $\displaystyle\lim_{x\to 0}\frac{1-\cos 3x}{\sin 3x} = \lim_{3x\to 0}\frac{1-\cos 3x}{3x}\cdot\frac{3x}{\sin 3x} = 0\cdot 1 = 0.$

12. $\displaystyle\lim_{x\to 0}\frac{\cos(x+\frac{\pi}{2})}{x} = \lim_{x\to 0}\frac{\sin(-x)}{x} = -1.$

13. By the chain rule $g'(x) = 2xf'(x^2)$, so $g'(-1)+g'(0)+g'(1) = 2(-1)f'(1)+2(0)f'(0)+2(1)f'(1) = -4+0+4 = 0$.

14. $V = \dfrac{4}{3}\pi r^3$, $\dfrac{dV}{dr} = 4\pi r^2$, $S = 4\pi r^2$, and $\dfrac{dS}{dr} = 8\pi r$. So, $\dfrac{dV}{dS} = \dfrac{dV/dr}{dS/dr} = \dfrac{r}{2} = \dfrac{t^2}{2}$. Thus $\dfrac{d}{dt}\left[\dfrac{dV}{dS}\right] = t$, and when $r=3$, $t=\sqrt{3}$.

15. Solving for y yields $y = \pm\sqrt{\dfrac{x+1}{x-1}}$. So, $\ln|y| = \dfrac{1}{2}\ln|x+1| - \dfrac{1}{2}\ln|x-1|$, and $\dfrac{d}{dx}[\ln|y|] = \dfrac{1}{2}\left(\dfrac{1}{x+1}-\dfrac{1}{x-1}\right) = \dfrac{1}{1-x^2}$.

16. Since $y^x = a$, $x\ln y = \ln a$. Using implicit differentiation, $\ln y + \dfrac{x}{y}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y}{x}\ln y$, and $e^{dy/dx} = y^{-y/x}$.

17. Differentiating implicitly, $2y\dfrac{dy}{dx} = 4$, or $\dfrac{dy}{dx} = \dfrac{2}{y}$. So, $\dfrac{d^2y}{dx^2} = -\dfrac{2}{y^2}\dfrac{dy}{dx} = -\dfrac{4}{y^3}$.

18. $f(x) = \dfrac{1}{x^2+x} = \dfrac{1}{x} - \dfrac{1}{x+1}$, so $f^{(n)}(x) = (-1)^n n!\left(\dfrac{1}{x^{n+1}} - \dfrac{1}{(x+1)^{n+1}}\right)$.

19. $f'(x) = D_x\left[\frac{(x+1)(x+2)}{(x-1)(x-2)}\right] = \frac{(x-1)(x-2)(2x+3)-(x+1)(x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2}$, which, when evaluated at $x=\sqrt{2}$ yields $f'(\sqrt{2})=0$.

20. $f(g(5)) = h(5)k(5)$, so $k(5)=2$. Now, differentiating $f(g(x)) = h(x)k(x)$ twice with respect to x yields $f''(g(x))[g'(x)]^2 + f'(g(x))g''(x) = h''(x)k(x) + h'(x)k'(x) + h'(x)k'(x) + h(x)k''(x)$. Substituting table values into that equation gives $6\times 16 + 3\times 1 = -5\times 2 + 2h'(5)k'(5) + 7\times 9$, so $h'(5)k'(5) = 23$.

21. $\lim_{x\to 0}\frac{\sqrt[3]{x}+2/\sqrt[3]{x}}{\sqrt[3]{x}-4/\sqrt[3]{x}} = \lim_{x\to 0}\frac{\sqrt[3]{x^2}+2}{\sqrt[3]{x^2}-4} = \frac{2}{-4} = -\frac{1}{2}$.

22. Clearly f is differentiable everywhere except possibly at $x=1$. The "derivative from the left" is $3x^2+2$ and the "derivative from the right" is k, so, at $x=1$, k would have to equal 5. But if $k=5$, then f is not continuous at $x=1$, so there is a contradiction of the assumption (that there was such a value for k), and no such value exists.

23. This is known as the Newton-Raphson method. Since $f'(x) = 2\cos x - 1$, we have $x_{n+1} = x_n - \frac{2\sin(x_n)-x_n}{2\cos(x_n)-1}$. Using this, and rounding each result, we compute starting with $x_0 = 2$ and obtain $x_1 = 1.90100$, $x_2 = 1.89551$, and finally $x_3 = 1.89549$.

24. The limit is the derivative of $[f(x)]^2$, which by the chain rule is $2f(x)f'(x)$.

25. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{1}{n}}{1+\frac{i}{n}} = \int_1^2 \frac{1}{x}\,dx = \ln 2$.

26. Let $u = \sqrt[3]{2\sin x}$. Then, using the chain rule and product rule, all the addends of the $3n$th derivative would contain a power of $(1-u)$ except for one, and that addend would be $(3n)!\left(-\frac{du}{dx}\right)^{3n}$. Since when $x=\frac{\pi}{6}$ we have $(1-u)=0$, all of the other addends drop out of our sum. Now, $\frac{du}{dx} = \frac{2\cos x}{3\sqrt[3]{4\sin^2 x}}$, which at $x=\frac{\pi}{6}$ gives $\frac{du}{dx} = \frac{1}{\sqrt{3}}$, so the answer is $\frac{(3n)!}{(-\sqrt{3})^{3n}}$ evaluated at $n=1$, which yields $-\frac{2\sqrt{3}}{3}$.

27. $y = 1+\frac{x}{y}$, so $x = y^2 - y$, $\frac{dx}{dy} = 2y-1$, and $\frac{dy}{dx} = \frac{1}{2y-1} = \frac{y}{2y^2-y} = \frac{y}{2y^2-2y+y} = \frac{y}{2x+y}$.

28. Simply taking the logarithm and differentiating shows that both I and II are true (they are known as logarithmic differentiation rules), whereas III can be seen to be true by noting that $h(a)k(a) = [f(a)]^2 > 0$ and $h(a)/k(a) = [g(a)]^2 > 0$.

29. Using the fact that $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$, we have $\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0}\frac{\frac{f(x+h)-f(x)}{h}-\frac{f(x)-f(x-h)}{h}}{h} = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = f''(x)$.

30. $\lim_{x\to a}\frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(\sqrt{x}+\sqrt{a}) = f'(a)\cdot 2\sqrt{a} = 2f'(a)\sqrt{a}$.