Complex Numbers (Alpha) — Solutions

FAMAT State Convention 2003

1. $\ln|3e^{5i+2} \times 7e^{-2i}| = \ln\left(|21e^2| \times |e^{3i}|\right) = \ln(21e^2) = 2 + \ln 21$.
2. Since it is simply the locus of all points a distance 3 from a given point $(-4+2i)$, the graph is a circle.
3. The angle measured from the x-axis up to $1+2i$ is $\arctan 2$, and the angle measured from the x-axis down to $1-3i$ is $\arctan 3$, so the angle between the two vectors is $\arctan 2 + \arctan 3 = \frac{3\pi}{4}$.
4. $r \operatorname{cis}\theta = \operatorname{cis} 51^\circ + \operatorname{cis} 65^\circ = (\cos 51^\circ + \cos 65^\circ) + i(\sin 51^\circ + \sin 65^\circ) = 2\cos 7^\circ \cos 58^\circ + 2i \cos 7^\circ \sin 58^\circ = 2\cos 7^\circ \operatorname{cis} 58^\circ$, so θ could equal 58°.
5. Since $\frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{2i}{2} = i$, the answer is simply $i^{2003} = -i$.
6. If $z = a+bi$ (with a and b real), then $\sqrt{z\overline{z}} = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2} = |z|$.
7. Simple arithmetic yields $-\frac{223}{1885} - \frac{3546}{1885}i$.
8. $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, so $ab = (a+b)^2 = a^2+2ab+b^2$, or $a^2+ab+b^2 = 0$. Dividing through by b^2 yields $z^2+z+1=0$, which has the solutions $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$.
9. I is only necessarily true for polynomials with only real coefficients. II and III are both true.
10. $\left(1+i\sqrt{3}\right)^{2003} + \left(1-i\sqrt{3}\right)^{2003} = (2\operatorname{cis} 60^\circ)^{2003} + (2\operatorname{cis} 300^\circ)^{2003} = 2^{2003}\left[\operatorname{cis}(2003 \times 60^\circ) + \operatorname{cis}(2003 \times 300^\circ)\right] =$ $2^{2003}(\operatorname{cis} 300^\circ + \operatorname{cis} 60^\circ) = 2^{2003}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2} + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 2^{2003}$.
11. All such numbers are exactly the roots of the polynomial $z^6 - z^3 - (6+5i)$, which has $A = 6$ distinct roots, the sum of which are $B = 0$, and the product being $C = -6-5i$. Thus, $A+B+C = -5i$.
12. $\frac{1}{z} + \frac{1}{\overline{z}} = \frac{\overline{z}+z}{z\overline{z}} = \frac{2\Re(z)}{|z|^2} = \frac{2(1)}{2^2} = \frac{1}{2}$.
13. $\frac{1}{2} + \frac{i\sqrt{3}}{2}$ is not an eighth root of unity since $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^8 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \neq 1$.
14. $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{2003} = (\operatorname{cis} 120^\circ)^{2003} = \operatorname{cis}(2003 \times 120^\circ) = \operatorname{cis} 240^\circ = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$.
15. Let $w = f(z) = \frac{\overline{z}}{\overline{z}-2}$. Then $\overline{w} = \frac{z}{z-2}$, and solving for z yields $z = \frac{2\overline{w}}{\overline{w}-1}$. Thus, $f^{-1}(z) = \frac{2\overline{z}}{\overline{z}-1}$.
16. We want the sum of the roots of $z^9 - (2-3i) = 0$, which is clearly 0.
17. $\arg\left(\frac{3+5i}{7+11i}\right) = \arg(3+5i) - \arg(7+11i) = \arctan\frac{5}{3} - \arctan\frac{11}{7} \approx 0.0263097 + 2k\pi$, which, taking $k=0$, is 0.0263 (correct to four decimal places).
18. $g(2+3i) = -1+2i$ and $f(-1+2i) = -7+8i$, so $f(g(2+3i)) = -7+8i$.

19. $|7+3i| = \sqrt{7^2+3^2} = \sqrt{58}$.

20. Let $z = a+bi$ (with a and b real). Then on the RHS all imaginary terms vanish, while on the LHS you get a term $2abi$; thus, either $a=0$ or $b=0$ (or both). $a=0 \implies -b^2 = |b|+b \implies b \le 0 \implies -b^2 = -b+b \implies b=0$. Meanwhile, $a \neq 0 \implies b=0 \implies a^2 = 3a+|a| \implies a>0 \implies a^2 = 4a \implies a=4$. Thus there are two solutions, $z=0$ and $z=4$.

21. Any nth degree polynomial with complex coefficients has exactly n complex roots (with possible multiple roots). It is simple to verify that 1 and 2 are roots, leading to i and $-i$ as the other two roots. Thus there are indeed 4 distinct complex roots.

22. Since the LHS of the equation is real, so must the RHS be; thus, z is purely imaginary, and $x=0$. Substituting in $z=iy$ yields $|3+iy| = 1+y$. Squaring both sides and solving for y gives $y=4$.

23. Since $|z| = \sqrt{z\overline{z}}$, $\left|\dfrac{2e^{i\theta}-i}{ie^{i\theta}+2}\right| = \sqrt{\dfrac{2e^{i\theta}-i}{ie^{i\theta}+2} \times \dfrac{2e^{-i\theta}+i}{-ie^{-i\theta}+2}} = \sqrt{\dfrac{5+2i(e^{i\theta}-e^{-i\theta})}{5+2i(e^{i\theta}-e^{-i\theta})}} = 1$.

24. I is true (simply consider when $a=b=2$ and $c=\dfrac{1}{2}$). II is false since the radical is used only to denote the principle root, which here would be $i\sqrt{|a|}$. III is true (should be basic knowledge).

25. If $z = x+iy$, where x and y are real, then $2 = z^2+\overline{z}^2 = x^2-y^2+2ixy+x^2-y^2-2ixy = 2x^2-2y^2$, the graph of which is a hyperbola.

26. Cross-multiply to get $z(1-it) = 1+it$, then solve for $it = \dfrac{z-1}{z+1}$, from which it is apparent that z cannot be -1.

27. Since $c-i = z+w = (a+i)+(3+ib) = (a+3)+(1+b)i$, $b=-2$ and $c=a+3$. Similarly, since $17+id = zw = (a+i)(3+ib) = (3a-b)+i(3+ab)$, $3a-b=17$ and $d=3+ab$. Along with $b=-2$ we solve and find that $a=5$ and $d=-7$, the answer.

28. $\displaystyle\sum_{n=1}^{2003}(ni^n) = \sum_{k=0}^{500}\left[(4k+1)i+(4k+2)i^2+(4k+3)i^3+(4k)i^4\right] = \sum_{k=0}^{500}[4ki+i-4k-2-4ki-3i+4k] = \sum_{k=0}^{500}(-2-2i) =$ $501(-2-2i) = -1002-1002i$.

29. $(-e)^{i\pi} = e^{i\pi\ln(-e)} = e^{i\pi[1+i\pi(2k+1)]}$, where k is any integer. Taking the $k=-1$ branch yields $e^{i\pi(1-i\pi)} = e^{i\pi}e^{-i^2\pi^2} = -e^{\pi^2}$

30. $(-8)^{\sqrt{3}} = e^{\sqrt{3}\ln(-8)} = e^{\sqrt{3}\ln 8+i\pi\sqrt{3}(2k+1)} = 8^{\sqrt{3}}\operatorname{cis}\left[\pi\sqrt{3}(2k+1)\right]$, so the argument is never a multiple of π (if it were, then $\sqrt{3}$ would be rational), and hence the number is never real. Further, no two arguments ever differ by a multiple of 2π (also based on the irrationality of $\sqrt{3}$), and hence there are infinitely many different values.