SEQUENCES AND SERIES – Calculus Division
FAMAT State Convention - 2003

.For all questions, e. NOTA means none of the above answers is correct.

1. Simplify $\left(1-\frac{3}{4}\right)\left(1-\frac{3}{5}\right)\left(1-\frac{3}{6}\right)\ldots\left(1-\frac{3}{n}\right)$

a. $\frac{6}{n(n-1)}$
b. $\frac{2}{n(n-1)}$
c. $\frac{1}{n(n-1)}$
d. $\frac{6}{n(n-1)(n-2)}$
e. NOTA

2. If $\lim_{n\to\infty} a_n = \frac{1}{4}$ then $\sum_{n=1}^{\infty} 3a_n$

a. converges to 3/4.
b. converges to 1/4.
c. converges to $3\left(\frac{1}{4}\right)^{n-1}$.
d. diverges
e. NOTA

3. Find the sum of the numerator and denominator after converting 0.31232323... into a fraction whose numerator and denominator are relatively prime positive integers.

a. 1458
b. 3248
c. 12992
d. 13023
e. NOTA

4. The power series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ is for which function?

a. e^x
b. $\frac{e^x}{x}$
c. $\frac{e^x - 1}{x}$
d. $\frac{e^x - 2}{x}$
e. NOTA

5. A ball is dropped from 100 feet off the ground and on each bounce it rebounds 1/2 the distance from which it fell. How far in feet up and down will it travel before coming to rest?

a. 300
b. 275
c. 250
d. 200
e. NOTA

6. If two six-sided dice (with faces numbered 1 to 6) are rolled, what is the probability of obtaining a sum of 7 before a sum of 11?

a. $\frac{1}{108}$
b. $\frac{23}{36}$
c. $\frac{2}{3}$
d. $\frac{3}{4}$
e. NOTA

7. Given that $\ln\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + ..,$ which of the following is equivalent to $\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + ... + \frac{1}{2^n \cdot n} + ...$?

a. $\frac{\sqrt{2}}{2}$
b. $\ln 2$
c. $\frac{\pi}{4}$
d. 0.7
e. NOTA

8. Find the product of the 3 positive geometric means between $\sqrt{2}$ and $8\sqrt{2}$.

a. $16\sqrt{2}$
b. 32
c. $32\sqrt{2}$
d. $64\sqrt{2}$
e. NOTA

9. Which sequences converge?

I. $\left\{\frac{\pi^n}{3^n}\right\}$

II. $\left\{\sqrt{n+2} - \sqrt{n}\right\}$

III. $\left\{\frac{\ln(2+e^n)}{3n}\right\}$

a. II only
b. II and III only
c. I and III only
d. III only
e. NOTA

10. $\lim_{n\to\infty} \frac{1}{n^{14}}\left(1^{13} + 2^{13} + 3^{13} + ... + n^{13}\right) =$

a. 1
b. 1/13
c. 1/14
d. 0
e. NOTA

11. Find all values of x for which the sequence $a_k = (\ln x)^k$ converges as $k \to \infty$.

a. $[1, e)$
b. $(0, e]$
c. $\left(\frac{1}{e}, e\right]$
d. $\left(\frac{1}{e}, e\right)$
e. NOTA

12. Suppose $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{3^n n!}$ for all values of x in the interval of convergence. Given that $-\frac{1}{2}$ is in the interval of convergence, find the maximum error involved if the first 2 terms of the power series are used to evaluate $f\left(-\frac{1}{2}\right)$.

a. 1/6
b. 1/12
c. 1/36
d. 1/48
e. NOTA

13. Find the first three non-zero terms in the Maclaurin series expansion for $f(x)=\sqrt{1+x^2}$.

a. $1+\frac{x}{2}-\frac{x^2}{8}$
b. $1+\frac{x^2}{2}-\frac{x^4}{8}$
c. $1+\frac{x^2}{2}+\frac{x^4}{4}$
d. $1+\frac{x^2}{2}-\frac{x^4}{4}$
e. NOTA

14. Suppose $x_0=1$ and $x_{n+1}=\frac{1}{2}\left(x_n+\frac{2}{x_n}\right)$ for $n=0,1,2,3,\ldots$. Let $\{x_n\}$ denote the sequence generated by the above recursive definition. Evaluate the limit of the sequence $\{x_n\}$ as n approaches infinity.

a. $\sqrt{2}$
b. $\sqrt{3}$
c. 1.42
d. $\frac{\sqrt{10}}{2}$
e. NOTA

15. Find the limit of the sequence $\left(\sqrt{2}-\sqrt{3}\right),\left(\sqrt{3}-\sqrt{4}\right),\left(\sqrt{4}-\sqrt{5}\right),\ldots$.

a. 0
b. $-\frac{1}{4}$
c. $-\frac{1}{2}$
d. -1
e. NOTA

16. The sequence 1,1,2,3,5,8,13,21,... is called the Fibonacci Sequence. Denoting the above sequence $\{a_n\}$, to what number does the sequence $\left\{\frac{a_{n+1}}{a_n}\right\}$ converge?

a. $\frac{1+\sqrt{5}}{2}$
b. 1.6
c. $\frac{\sqrt{5}-1}{2}$
d. $\frac{5}{8}$
e. NOTA

17. $\sum_{k=0}^{\infty} ar^k$ always converges for which values of r?

a. $r<1$
b. $|r|<1$
c. $|r|\le 1$
d. $|r|>1$
e. NOTA

18. For all real values of x,
$\sin x-\frac{1}{2}\sin^2 x+\frac{1}{4}\sin^3 x-\ldots=$

a. $\frac{2\sin x}{2-\sin x}$
b. $\frac{\sin x}{1-\sin x}$
c. $\frac{2\sin x}{2+\sin x}$
d. $\frac{2\sin x}{1+\sin x}$
e. NOTA

19. Evaluate $\sum_{k=0}^{11}(5+2k)$

a. 162
b. 176
c. 192
d. 204
e. NOTA

20. Find the 3rd degree Taylor polynomial for $\ln x$ centered at $x=2$.

a. $\ln 2+\frac{x-2}{2}-\frac{(x-2)^2}{8}+\frac{(x-2)^3}{24}$
b. $\ln 2+\frac{x-2}{2}-\frac{(x-2)^2}{4}+\frac{(x-2)^3}{4}$
c. $\frac{x-2}{2}-\frac{(x-2)^2}{8}+\frac{(x-2)^3}{24}$
d. $\ln 2+\frac{x-2}{2}-\frac{(x-2)^2}{8}+\frac{(x-2)^3}{12}$
e. NOTA

21. Which of the following series converge?

I. $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$

II. $\sum_{k=1}^{\infty}\frac{2}{2\sqrt{k}+1}$

III. $\sum_{k=1}^{\infty}\frac{1}{k}$

a. I, II, and III
b. I and II only
c. I only
d. II only
e. NOTA

22. $\sum_{k=1}^{\infty}3^{2k}\cdot 5^{1-k}$ =

a. 10
b. 43/4
c. 11
d. 45/4
e. NOTA

23. What is the sum of the first 50 consecutive even <u>whole numbers</u>?

a. 2550
b. 2450
c. 2350
d. 2250
e. NOTA

24. Find the positive geometric mean between $2x$ and $8x$.

a. $4x$
b. $4|x|$
c. $8x$
d. $18x$
e. NOTA

25. An equilateral triangle has a side of length 16. Line segments joining the midpoints form a second triangle. Then line segments joining midpoints of the second triangle form a third triangle and so on. What is the limiting value of the sum of the areas of all of these equilateral triangles?

a. $\frac{64\sqrt{3}}{3}$
b. $\frac{256\sqrt{3}}{3}$
c. $64\sqrt{3}$
d. $256\sqrt{3}$
e. NOTA

26. $\sum_{k=0}^{10}\binom{10}{k} =$

a. 1024
b. 512
c. 256
d. 128
e. NOTA

27. Which of the following converge conditionally but not absolutely?

I. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

II. $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n}$

III. $\sum_{n=2}^{\infty}\frac{(-1)^{n}}{n \cdot \ln n}$

a. I, II, and III
b. I only
c. I and III only
d. II and III only
e. NOTA

28. Evaluate $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...$

a. $\frac{\pi}{6}$
b. $\frac{\sqrt{2}}{2}$
c. 0.7
d. $\ln 2$
e. NOTA

29. $\sum_{k=1}^{n}\ln k =$

a. $\ln(n!)$
b. $\ln\frac{n(n+1)}{2}$
c. $\ln n^{n}$
d. $\ln(n+1)$
e. NOTA

30. The sum of an infinite geometric series is 8. The sum of the cubes of its terms is $\frac{512}{7}$. What is its first term?

a. 6
b. 4.5
c. 4
d. 3.5
e. NOTA

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D 1. $\frac{1}{\not 4}\cdot\frac{2}{\not 5}\cdot\frac{3}{\not 6}\cdot\frac{\not 4}{\not 7}\cdot\frac{\not 5}{\not 8}\cdot\ldots\frac{\not{n-5}}{n-2}\cdot\frac{\not{n-4}}{n-1}\cdot\frac{\not{n-3}}{n}=\frac{6}{n(n-1)(n-2)}$

D 2. Then $\lim_{n\to\infty} 3a_n=\frac{3}{4}\neq 0$. The sum of terms will diverge by the nth term test.

B 3. 100x = 31.232323... and x = .31232323, so 99x = 30.92. $x=\frac{3092}{9900}=\frac{773}{2475}$

D 4. $e^x-1=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!};\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-2}}{(n-1)!}$. Subtracting 1/x from both sides gives $\frac{e^x-2}{x}$ which is equal to the given series.

A 5. $2\left(\frac{100}{1-.5}\right)-100=300$

D 6. Sum = $\frac{1}{6}+\frac{28}{36}\cdot\frac{1}{6}+\left(\frac{28}{36}\right)^2\cdot\frac{1}{6}+\ldots=\frac{\frac{1}{6}}{1-\frac{28}{36}}=\frac{3}{4}$

B 7. Just substitute x = 1/2 in both sides of the given equation.

E 8. $8\sqrt{2}=\sqrt{2}r^4; r^4=8; r^2=\sqrt{8}.$ The product of the 3 means is $\sqrt{2}r\cdot\sqrt{2}r^2\cdot\sqrt{2}r^3=\sqrt{8}r^6$. Substituting gives $\sqrt{8}\cdot\sqrt{8}^3=\sqrt{8}^4=64$

B 9. I. The limit is infinity. II. $\lim_{n\to\infty}\frac{\left(\sqrt{n+2}-\sqrt{n}\right)\left(\sqrt{n+2}+\sqrt{n}\right)}{\sqrt{n+2}+\sqrt{n}}=\lim_{n\to\infty}\frac{2}{\sqrt{n+2}+\sqrt{n}}=0$

III. Using L'Hopital's rule, the limit of a_n is 1/3.

C 10. This is $\int_0^1 x^{13}dx=\frac{1}{14}$.

C 11. The sequence is geometric with r equal to $\ln x$. It will converge when $-1<\ln x<1$ giving $\frac{1}{e}<x<e$. But it also converges for x = e since each term is equal to 1.

D 12. The series is alternating when $-1/2$ is substituted for x. The maximum error is the absolute value of the first term dropped which is $\frac{\left(-\frac{1}{2}\right)^3\cdot 27}{27\cdot 6}=-\frac{1}{48}$.

B 13. Use Taylor's formula $\sqrt{1+x}\approx 1+\frac{x}{2}-\frac{x^2}{8}$. Now substitute x^2 for x in both sides.

A 14. If $\lim_{n\to\infty}x_n=L$, then $\lim_{n\to\infty}x_{n+1}=L$ also. So $x_{n+1}=\frac{1}{2}\left(x_n+\frac{2}{x_n}\right)$. As $n\to\infty$ we obtain $L=\frac{1}{2}\left(L+\frac{2}{L}\right)$. Solving for L gives $\pm\sqrt{2}$ but the limit is obviously positive.

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A 15. $\lim_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n+2}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)}{\sqrt{n+1}+\sqrt{n+2}} = \lim_{n\to\infty} \frac{-1}{\sqrt{n+1}+\sqrt{n+2}} = 0$

A 16. The limit consecutive terms of the Fibonacci sequence is the golden ratio which is A.

B 17. This is a geometric series which converges which the absolute value of r is less than 1.

C 18. The series is geometric with r equal to $-\frac{1}{2}\sin x$ which is always between 1 and -1.

The sum is $\frac{\sin x}{1-\left(-\frac{1}{2}\sin x\right)} = \frac{2\sin x}{2+\sin x}$

C 19. $S_n = \frac{n}{2}\left(2a+d(n-1)\right) = \frac{12}{2}\left(10+22\right) = 192$

A 20. Use Taylor's formula with a = 2 to get choice A.

E 21. They all diverge by p-series or by limit comparison to a p-series.

E 22. This series is equivalent to $\sum_{k=1}^{\infty} 9\cdot\left(\frac{9}{5}\right)^{k-1}$ which is geometric with $r>1$ so it diverges.

B 23. 0 + 2 + 4 + ...+ 98 = $\frac{50}{2}(0+98) = 2450$

B 24. $m = \sqrt{ab} = \sqrt{16x^2} = 4|x|$

B 25. S = $\frac{64\sqrt{3}}{1-\frac{1}{4}} = \frac{256\sqrt{3}}{3}$. This series is geometric again with r = 1/4.

A 26. The sum of $\binom{n}{k}$ as k goes from 0 to n is always 2^n.

C 27. I. Converges conditionally by the Alternating Series Test. II. Converges absolutely since it is a geometric series. III. Converges by the Alternating Series Test but diverges when the Integral Test is used on the sum of the absolute values.

D 28. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$ using Taylor's Formula. Put x = 1 in both sides.

A 29. $\ln 1 + \ln 2 + \ln 3 + \ldots = \ln(1\cdot 2\cdot 3\ldots n) = \ln(n!)$

C 30. $\frac{a}{1-r} = 8$ and $\frac{a^3}{1-r^3} = \frac{512}{7}$. So $7a^3 = 512(1-r)(1+r+r^2)$. But using the first equation, $a^3 = 512(1-r)^3$. Substitute for a^3 to get $7\cdot 512(1-r)^3 = 512(1-r)(1+r+r^2)$. Simplify this last equation to get $7(1-r)^2 = 1+r+r^2$ or $2r^2 - 5r + 2 = 0$. There are two solutions, namely r = 1/2 and r = 2 but the series will not converge for r = 2. Substitute in the first equation to get a = 4.