

Limits & Derivatives (Calculus)

FAMAT State Convention 2003

Instructions

Choose the single correct answer to each question from the choices A–D. If no single correct answer exists for a particular question, choose E for None Of The Above (NOTA).

1. Evaluate: $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$.
A. 0 B. $\frac{1}{2}$ C. 1 D. ∞ E. NOTA
2. If $\lfloor x \rfloor$ is the greatest integer less than or equal to x , then which answer choice is equal to the product $\left(\lim_{x \rightarrow n^-} \lfloor x \rfloor \right) \left(\lim_{x \rightarrow n^+} \lfloor x \rfloor \right)$? (Assume n is an integer greater than 2.)
A. $1 + 2 + \cdots + n$ B. $n^2 + n$ C. n^2 D. $2 + 4 + \cdots + 2n$ E. NOTA
3. If $f(x) = (x - 1)^2(x + 2)^2 + 5x + 2$, then what is $D_x[f(1)]$?
A. 0 B. 5 C. 9 D. 14 E. NOTA
4. Using the ϵ - δ definition of the limit, what is the greatest value (in terms of ϵ) that one could use for δ to prove that $\lim_{x \rightarrow 10} (4x + 5) = 45$?
A. ϵ B. $\frac{\epsilon}{2}$ C. $\frac{\epsilon}{3}$ D. $\frac{\epsilon}{4}$ E. NOTA
5. $|g(x)| \leq M$ for a fixed M and for all $x \neq a$. If $\lim_{x \rightarrow a} f(x) = 0$, then what is $\lim_{x \rightarrow a} [f(x)g(x)]$?
A. 0 B. $\frac{M}{2}$ C. M D. Not enough information E. NOTA
6. Differentiate with respect to x : $\arctan\left(\frac{1}{x+1}\right)$.
A. $\frac{1}{|1+x|\sqrt{x^2+2x}}$ B. $\frac{-1}{|1+x|\sqrt{x^2+2x}}$ C. $\frac{1}{x^2+2x+2}$ D. $\frac{-1}{x^2+2x+2}$ E. NOTA

7. Evaluate: $\lim_{x \rightarrow -\infty} \frac{x+2}{\sqrt{x^2+4}}$.
- A. 0 B. $\frac{1}{2}$ C. 1 D. 2 E. NOTA
8. Evaluate: $\lim_{h \rightarrow 0} \frac{f(x+ah) - f(x+bh)}{h}$.
- A. $f'(x)$ B. $2f'(x)$ C. $(a+b)f'(x)$ D. $(a-b)f'(x)$ E. NOTA
9. Differentiate with respect to x : $\ln \sqrt{1 - \sin^2 x}$.
- A. $2 \tan x$ B. $-2 \tan x$ C. $\tan x$ D. $-\tan x$ E. NOTA
10. A particle is traveling along the x -axis. Its position at time $t \geq 0$ is given by $x = (3 - 2t)/(5t + 2)$. What is the acceleration of the particle when $t = 2$?
- A. $\frac{19}{144}$ B. $\frac{95}{864}$ C. $\frac{55}{864}$ D. $-\frac{19}{144}$ E. NOTA
11. Evaluate: $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{\sin 3x}$.
- A. 1 B. $\frac{1}{3}$ C. 0 D. -1 E. NOTA
12. Evaluate: $\lim_{x \rightarrow 0} \frac{\cos(x + \frac{\pi}{2})}{x}$.
- A. 1 B. 0 C. -1 D. Indeterminate E. NOTA
13. Let f be a differentiable function and let $g(x) = f(x^2)$. If $f'(1) = 2$ and $f'(-1) = 3$, then what is the sum $g'(-1) + g'(0) + g'(1)$?
- A. -2 B. 0 C. 2 D. Not enough information E. NOTA
14. Find $\frac{d}{dt} \left[\frac{dV}{dS} \right]$ when $r = 3$ if V is the volume enclosed by a sphere of radius r , S is the surface area of a sphere of radius r , and $r = t^2$, where $t \geq 0$ is the time according to which the sphere is expanding.
- A. $\frac{2\sqrt{3}}{3}$ B. $\sqrt{3}$ C. 2 D. 3 E. NOTA
15. $xy^2 - y^2 - x - 1 = 0$. Find $\frac{d}{dx} [\ln |y|]$.
- A. $\frac{1}{1-x^2}$ B. $\frac{1}{2-2x^2}$ C. $\frac{x+2}{1+x-2x^2}$ D. $\frac{x+3}{2-2x^2}$ E. NOTA

16. Let $y = f(x)$ be a differentiable function of x such that $[f(x)]^x = a$ (a is a constant greater than 1). What is $e^{dy/dx}$?

- A. $x^{-x/y}$ B. $e^{-x/y}$ C. $y^{-y/x}$ D. $e^{-y/x}$ E. NOTA

17. What is $\frac{d^2y}{dx^2}$ if $y^2 = 4x$?

- A. $-\frac{1}{4\sqrt{x^3}}$ B. $-\frac{2}{\sqrt{x^3}}$ C. $-\frac{2}{y^2}$ D. $-\frac{4}{y^3}$ E. NOTA

18. If $f(x) = \frac{1}{x^2 + x} = \frac{1}{x} - \frac{1}{x+1}$, then what is $f^{(n)}(x)$, the n th derivative of $f(x)$?

- A. $(-1)^n(n-1)! \left(\frac{1}{x^n} - \frac{1}{(x+1)^n} \right)$ B. $(-1)^{n-1}(n-1)! \left(\frac{1}{x^n} - \frac{1}{(x+1)^n} \right)$
 C. $(-1)^n n! \left(\frac{1}{(x+1)^n} - \frac{1}{x^n} \right)$ D. $(-1)^n n! \left(\frac{1}{x^{n+1}} - \frac{1}{(x+1)^{n+1}} \right)$ E. NOTA

19. What is $f'(\sqrt{2})$ if $f(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$?

- A. 0 B. 1 C. 2 D. Undefined E. NOTA

20. Given that $f(g(x)) = h(x)k(x)$ and that

$$\begin{array}{llll} f(5) = 14 & g(5) = 5 & h(5) = 7 & k(5) = \\ f'(5) = 3 & g'(5) = 4 & & \\ f''(5) = 6 & g''(5) = 1 & h''(5) = -5 & k''(5) = 9, \end{array}$$

first find $k(5)$. Then, by differentiating, determine the value of $h'(5)k'(5)$. What is this value?

- A. $\frac{11}{2}$ B. 18 C. 23 D. Not enough information E. NOTA

21. Evaluate: $\lim_{x \rightarrow 0} \frac{\sqrt[3]{x} + 2/\sqrt[3]{x}}{\sqrt[3]{x} - 4/\sqrt[3]{x}}$.

- A. $+\infty$ B. 1 C. 0 D. $-\infty$ E. NOTA

22. If

$$f(x) = \begin{cases} x^3 + 2x + 1 & \text{if } x \leq 1 \\ kx + 9 & \text{if } x > 1, \end{cases}$$

then for what number k is f differentiable everywhere?

- A. -5 B. 0 C. 5 D. 10 E. NOTA

23. Successive approximations of a zero of a function f are given by $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Starting with $x_0 = 2$, what approximation of the positive solution of $f(x) = 2 \sin x - x = 0$ would you obtain after 3 iterations (i.e. what's x_3)? Round all results (intermediate and final) to five decimal places.

- A. 0.00000 B. 0.10032 C. 1.89549 D. 1.89646 E. NOTA

24. Evaluate: $\lim_{h \rightarrow 0} \frac{[f(x+h)]^2 - [f(x)]^2}{h}$.

- A. $2f(x)$ B. $2f'(x)$ C. $2f(x)f'(x)$ D. $[f'(x)]^2$ E. NOTA

25. Evaluate: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n+i}$.

- A. 0 B. $\ln 2$ C. 1 D. ∞ E. NOTA

26. Evaluate at $x = \frac{\pi}{6}$: $\frac{d^3}{dx^3} \left[(1 - \sqrt[3]{2 \sin x})^3 \right]$.

- A. 0 B. 1 C. $-\frac{2\sqrt{3}}{3}$ D. $\frac{2}{9}$ E. NOTA

27. If $y = 1 + \frac{x}{y}$, then what is $\frac{dy}{dx}$?

- A. $\frac{1}{y}$ B. $\frac{1}{2y}$ C. $\frac{1}{2y+1}$ D. $\frac{y}{2x+y}$ E. NOTA

28. Suppose that $f'(a)$ and $g'(a)$ exist, that $f(a) \neq 0$ and $g(a) \neq 0$, and that $h(x) = f(x)g(x)$ and $k(x) = f(x)/g(x)$. Which of the following statements is(are) necessarily true? [Hint: If $a = b \times c$, then $\ln a = \ln b + \ln c$.]

I. $h'(a)/h(a) = f'(a)/f(a) + g'(a)/g(a)$.

II. $k'(a)/k(a) = f'(a)/f(a) - g'(a)/g(a)$.

III. Both $h(a)k(a)$ and $h(a)/k(a)$ are positive.

- A. I, II, & III B. I & II only C. I only D. III only E. NOTA

29. Evaluate: $\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$.

- A. $f''(x)$ B. $2f'(x)$ C. ∞ D. 0 E. NOTA

30. Evaluate: $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$.

- A. $f'(a)$ B. $f'(x)$ C. $2f'(a)\sqrt{a}$ D. $2f'(x)\sqrt{a}$ E. NOTA