Complex Numbers (Alpha)

FAMAT State Convention 2003

Instructions

Choose the single correct answer to each question from the choices A–D. If no single correct answer exists for a particular question, choose E for None Of The Above (NOTA). On this test, $i = \sqrt{-1}$, and if $z = a + bi$, where a and b are real, then $\Re(z) = a$ (the real part) and $\Im(z) = b$ (the imaginary part). Also, $|z|$ represents the distance from z to the origin, and the complex conjugate of z is represented by $\overline{z}$.

1. What is $\ln|3e^{5i+2} \times 7e^{-2i}|$?

   A. $2 + \ln 21 + 3i$ B. $2 + \ln 21$ C. $\sqrt{13} + \ln 21$ D. $\ln 21$ E. NOTA

2. Which answer choice describes the graph of $|z + 4 - 2i| = 3$ in the complex plane?

   A. one point B. two points C. a line D. a circle E. NOTA

3. What is the measure, in radians, of the smaller angle between the position vectors of $1+2i$ and $1-3i$ in the complex plane?

   A. $\frac{\pi}{3}$ B. $\frac{\pi}{2}$ C. $\frac{2\pi}{3}$ D. $\frac{3\pi}{4}$ E. NOTA

4. $r \operatorname{cis}\theta = \operatorname{cis} 51° + \operatorname{cis} 65°$, where r and θ are real. What could θ equal (in degrees)?

   A. 14° B. −14° C. 116° D. 58° E. NOTA

5. $\left(\frac{1+i}{1-i}\right)^{2003} =$

   A. $1+i$ B. $-i$ C. $-1+i$ D. i E. NOTA

6. Which answer choice is equivalent to $\sqrt{z\overline{z}}$?

   A. $\Re(z)$ B. $\Im(z)$ C. $\sqrt{[\Re(z)]^2 - [\Im(z)]^2}$ D. $|z|$ E. NOTA

7. If $x = 2 + 3i$, $y = 5i - 7$, and $z = 11 + 13i$, then what is $\frac{xy - yz}{zx}$?

   A. $\frac{3534}{1885} + \frac{367}{1885}i$ B. $-\frac{223}{1885} - \frac{3546}{1885}i$ C. $-\frac{564}{2005} - \frac{3557}{2005}i$ D. $\frac{1361}{1885} + \frac{3282}{1885}i$ E. NOTA

8. If $z = \frac{a}{b}$ and $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$, then what could z be?

A. $\frac{\sqrt{3}}{2} - \frac{i}{2}$ B. $-\frac{\sqrt{3}}{2} + \frac{i}{2}$ C. $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ D. Not enough information E. NOTA

9. Which of the following statements is(are) true?

I. If $2 + 3i$ is a root of a polynomial equation, then so is $2 - 3i$.

II. Every polynomial with real coefficients and of odd degree has at least one real root.

III. All polynomials with real coefficients have an even number of roots that are not real.

A. I & II only B. I & III only C. II & III only D. I, II, & III E. NOTA

10. Simplify: $\left(1 + i\sqrt{3}\right)^{2003} + \left(1 - i\sqrt{3}\right)^{2003}$.

A. $2^{2003}\left(1 + i\sqrt{3}\right)$ B. $2^{2003}\left(1 - i\sqrt{3}\right)$ C. 2^{2003} D. 2^{2004} E. NOTA

11. When the complex number z is raised to the third power it is $6 + 5i$ less than when it is raised to the sixth power. Let A be the number of complex numbers with this property, B be the sum of all such numbers, and C be the product of all such numbers. What is $A + B + C$?

A. $-5i$ B. $12 + 5i$ C. $12 - 5i$ D. Not enough information E. NOTA

12. If $\Re(z) = 1$ and $|z| = 2$ then what is the value of $\frac{1}{z} + \frac{1}{\overline{z}}$?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. Not enough information E. NOTA

13. Which answer choice is not an eighth root of unity?

A. 1 B. $-i$ C. $\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$ D. $\frac{1}{2} + \frac{i\sqrt{3}}{2}$ E. NOTA

14. Evaluate: $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{2003}$.

A. $\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$ B. $\frac{\sqrt{3}}{2} + \frac{i}{2}$ C. $-\frac{\sqrt{3}}{2} + \frac{i}{2}$ D. $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ E. NOTA

15. If $f(z) = \frac{\overline{z}}{\overline{z} - 2}$, then what is its inverse, $f^{-1}(z)$?

A. $\frac{2\overline{z}}{\overline{z} - 1}$ B. $\frac{2\overline{z}}{1 - \overline{z}}$ C. $\frac{\overline{z} - 1}{2\overline{z}}$ D. $\frac{1 - \overline{z}}{2\overline{z}}$ E. NOTA

16. Which answer choice is equal to the sum of the ninth roots of $2 - 3i$?

A. -1 B. 0 C. 1 D. $-2 + 3i$ E. NOTA

17. Which answer choice could be the argument of $\dfrac{3+5i}{7+11i}$, in radians, correct to four decimal places?

A. -0.0263 B. 0.0263 C. 0.4472 D. 1.0262 E. NOTA

18. If $f(z) = z^2 + 5z + 1 + 2i$ and $g(z) = \dfrac{z+3i}{4+i-z}$, then what is $f(g(2+3i))$?

A. $-1 + 2i$ B. $-\dfrac{-227}{197} + \dfrac{26}{197}i$ C. $6 + 29i$ D. $-7 + 8i$ E. NOTA

19. What is the modulus of $7 + 3i$?

A. 10 B. 21 C. $\sqrt{10}$ D. $\sqrt{21}$ E. NOTA

20. $z^2 = z + \overline{z} + |z| + \Re(z) + \Im(z)$ has two solutions. The first is $z = 0$. What is the second one?

A. -4 B. $3 - 4i$ C. $4 - 3i$ D. 4 E. NOTA

21. How many distinct complex roots does the equation $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$ have?

A. 0 B. 1 C. 2 D. 3 E. NOTA

22. What ordered pair, (x, y), of real numbers is a solution of $|z + 3| = 1 - iz$, if $z = x + iy$?

A. $(3, 1)$ B. $(2, 2)$ C. $(1, 3)$ D. $(0, 4)$ E. NOTA

23. Find $\left|\dfrac{2e^{i\theta} - i}{ie^{i\theta} + 2}\right|$.

A. $|2\cos\theta|$ B. $|2\cos 2\theta|$ C. $|2\sin 2\theta|$ D. Not enough information E. NOTA

24. If a, b, and c are all complex numbers different from 0, then which of the following statements are true?

I. $(a^b)^c$ can have more values than $a^{(bc)}$.

II. $\sqrt{-|a|}$ has exactly two values.

III. $(b)^{\frac{1}{n}}$, where n is a positive integer, has exactly n values.

A. I & II only B. I & III only C. II & III only D. I, II, & III E. NOTA

25. Which answer choice describes the graph of $z^2 + \overline{z}^2 = 2$ in the complex plane?

A. one point **B. a line** **C. a hyperbola** D. a circle E. NOTA

26. Any complex number z with $|z| = 1$ can be expressed in the form $z = \dfrac{1+it}{1-it}$ with the appropriate choice of the real parameter t, with the only exception being w, which cannot be so expressed. What is w?

A. i B. $-i$ C. 1 D. -1 E. NOTA

27. If a, b, c, and d are real numbers such that

$$z = a + i \qquad w = 3 + ib$$
$$z + w = c - i \qquad zw = 17 + id$$

are all true, then what is the value of d?

A. -2 B. 5 C. -7 D. 8 E. NOTA

28. Evaluate: $\displaystyle\sum_{n=1}^{2003} (ni^n)$.

A. $1002 + 1002i$ B. $1002 - 1002i$ C. $-1002 + 1002i$ D. $-1002 - 1002i$ E. NOTA

29. Which answer choice is a value for $(-e)^{i\pi}$?

A. $-e^{\pi^2}$ B. -1 C. 1 D. ie^{π} E. NOTA

30. Fill in the blanks: $(-8)^{\sqrt{3}}$ has a(n) _____ number of values, _____ of which is(are) real

A. finite; none B. finite; one C. infinite; none D. infinite; one E. NOTA