BC Calculus Topic Test Solutions – FAMAT State Convention 2003

1. $\int_0^{\pi} \cos^2 \lambda \, d\lambda = \int_0^{\pi} \left( \frac{1+\cos(2\lambda)}{2} \right) d\lambda = \frac{\lambda}{2} + \frac{\sin(2\lambda)}{4} \Big]_0^{\pi} = \frac{\pi}{2}$. **B**

2. $(0,2)$ occurs when $t=0$. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - \ln(t+1) - 1}{2t\cos t - t^2 \sin t + 2}$. When $t=0$, $\frac{dy}{dx} = -\frac{1}{2}$. **B**

3. $\frac{\sin x \cdot \cos x}{x}$ is indeterminate and L'Hospital's Rule must be used.

$\lim_{x\to 0} \frac{\sin x \cdot \cos x}{x} = \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x}{1} = 1 - 0 = 1$. **C**

4. $y' = \frac{-y}{2x}$. $y_{n+1} = y_n - \frac{y_n}{2x_n} \cdot \Delta x$. $\Delta x = .1$ and $x_0 = y_0 = 1$.

$y_1 = y_0 - \frac{y_0}{2x_0} \cdot \Delta x = 1 - \frac{1}{2 \cdot 1} \cdot .1 = .95$. $y_2 = y_1 - \frac{y_1}{2x_1} \cdot \Delta x = .95 - \frac{.95}{2 \cdot 1.1} \cdot .1 \approx .907 \approx y(1.2)$. **B**

5. $\lim_{x\to\infty} \frac{n^2}{n^2+1} = 1$, thus I diverges. $\frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$ Since $\frac{1}{2} < 1$, II diverges. III converges by the ratio test. Only III converges. **A**

6. $(2y-6)\,dy = (8x+8)\,dx$. Integrating both sides gives $y^2 - 6y = 4x^2 + 8x + C$. $(y-3)^2 - 9 = 4(x+1)^2 - 4 + C$. $(y-3)^2 = 4(x+1)^2 + 5 + C$. The particular solution yielding two intersecting lines occurs when $C = -5$. $(y-3)^2 = 4(x+1)^2$ or $\pm(y-3) = \pm 2(x+1)$. The two lines are $y_1 = 2x+5$ and $y_2 = -2x+1$. $y_1(1) = 7$ and $y_2(1) = -1$. $7 - 1 = 6$. **D**

7. Use the limit comparison test with $\frac{1}{n}$.

$\lim_{n\to\infty} \frac{\frac{n^2 - n + n^6 - e^4 n^3}{\pi n^3 - 6n^4 + n^7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3 - n^2 + n^7 - e^4 n^4}{\pi n^3 - 6n^4 + n^7} = 1 > 0$. Since $\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty} \frac{n^2 - n + n^6 - e^4 n^3}{\pi n^3 - 6n^4 + n^7}$ diverges. **D**

8. The arc length is given by $\int_0^3 \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt = \int_0^3 3\, dt = 9$. **E**

9. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ $\quad \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$

$\int \frac{\sin x}{x} dx = \int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \ldots + C$ **D**

10. The substitution made is $\tan\theta = 2x$ and $\sec\theta = \sqrt{4x^2 + 1}$. The lower limit is $\arctan 0 = 0$ and the upper limit is $\arctan\sqrt{3} = \frac{\pi}{3}$. $0 + \frac{\pi}{3} = \frac{\pi}{3}$. **C**

11. Let $u = x$ and $dv = \cos x\, dx$. $du = dx$ and $v = \sin x$.

$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C$. **A**

12. The limit is indeterminate. Using L'Hospital's rule with the Second Fundamental Theorem of Calculus, the limit becomes

$\lim_{x\to\infty}\left(\sqrt{x^2 + 5x} - x\right) = \lim_{x\to\infty}\left(\sqrt{x^2 + 5x + 6.25 - 6.25} - x\right) = \lim_{x\to\infty}\left(\sqrt{(x + 2.5)^2 - 6.25} - x\right) = 2.5$

**C**

13. $ds = \sqrt{1 + (y')^2}\, dx$. $y' = -\sin x$. $ds = \sqrt{1 + \sin^2 x}\, dx$. Let $x = \frac{\pi}{4}$ and $dx = \frac{\pi}{4}$.

$ds = \sqrt{1 + \sin^2 \frac{\pi}{4}} \cdot \frac{\pi}{4} \approx .96$. **C**

14. $\frac{d}{dx}\left(\arccos\frac{1}{x}\right) = -\frac{\frac{-1}{x^2}}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} = \frac{1}{x^2\sqrt{\frac{x^2 - 1}{x^2}}} = \frac{1}{|x|\sqrt{x^2 - 1}}$. $\frac{\sqrt{3}}{6}$ **B**

15. The area of the rug is given by

$\int_0^\pi (12 + 12\cos\theta)^2\, d\theta = \int_0^\pi \left(144 + 288\cos\theta + 144\cos^2\theta\right) d\theta =$

**B**

$\int_0^\pi \left(216 + 282\cos\theta + 72\cos(2\theta)\right) d\theta = 216\theta + 282\sin\theta + 36\sin(2\theta)\Big]_0^\pi = 216\pi$

16. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \ldots$, and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} - \ldots$.

$\cos i = 1 - \frac{i^2}{2!} + \frac{i^4}{4!} - \frac{i^6}{6!} + \frac{i^8}{8!} - \ldots = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \ldots$,

$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \ldots$, and

$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \ldots$. $e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \frac{2}{8!} + \ldots = \frac{\cos i}{2}$.

$\cos i = \frac{e + e^{-1}}{2} = \frac{e^2 + 1}{2e}$. **A**

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the Alternating Series Test. However, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ is the harmonic series, and diverges by the *p*-series Test. As a result, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent. One property of a conditionally convergent series is that the terms can be rearranged in a certain manner such that the series can converge to any desired sum or diverge. As a result, I and II are true statements, while III and IV are false. **B**

18. $\theta = \frac{\pi}{6}$ so $r = \frac{\sqrt{3}}{2}$. $r'(\theta) = -\sin\theta$ and $r'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$. $x = r\cos\theta$ and $y = r\sin\theta$.

$\frac{dx}{d\theta} = r'(\theta)\cos\theta - r(\theta)\sin\theta$ and $\frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta$.

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$. When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{-\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}{-\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}} = -\frac{\sqrt{3}}{3}$. **A**

19. First note that choices B and C are equivalent. $dy = \tan\theta\sec^2\theta\, d\theta$. Integrating with substitutions of $u = \sec\theta\tan\theta$ and $u = \tan\theta$ will yield choices C and D, respectively. B, C, and D satisfy the differential equation while A does not. **A**

20. Using L'Hospital's Rule, $\lim_{x\to 0^+} x^3 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-3}{x^4}} = \lim_{x\to 0^+} \frac{-x^3}{3} = 0$. **B**

21. $\int_0^{\frac{\pi}{2}} 5\sin^3 x\cos^3 x\,dx = \int_0^{\frac{\pi}{2}} 5\sin^3 x\left(1-\sin^2 x\right)\cos x\,dx = \int_0^{\frac{\pi}{2}}\left(5\sin^3 x - 5\sin^5 x\right)\cos x\,dx =$

$\left(\frac{5\sin^4 x}{4} - \frac{5\sin^6 x}{6}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{5}{4} - \frac{5}{6} = \frac{5}{12}$. $5 + 12 = 17$. **D**

22. Let $(x_0, y_0)$ be any point on the original graph. $\frac{dy}{dx}$ at $(x_0, y_0)$ is the slope of $y_1$, the line tangent to the graph at $(x_0, y_0)$. If the graph is rotated $45°$ clockwise, $(x_0, y_0)$ will be transformed to $(x_0', y_0')$, and $\frac{dy}{dx}$ at this point will be the slope of the line $y_1'$. The point $\left(\sqrt{2}, 0\right)$ is a point on the rotated graph, and this point can under go a rotation transformation to yield to corresponding point on the original graph.
$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The derivative of the original graph is

$\frac{dy}{dx} = \frac{-\left(9x^2 - 8xy - 3\right)}{-4x^2 + 12y - 6}$. The slope of the line tangent at $(1,1)$ is therefore 1. If a line with slope 1 is rotated $45°$ clockwise, the new line will be horizontal, and thus the slope is 0.
**B**

23. $\int_0^{a}\left(\cos^4 x - \sin^4 x\right)dx = \int_0^{a}\left[\left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right)\right]dx = \int_0^{a}\cos(2x)\,dx$

$\frac{\sin(2x)}{2}\Bigg|_0^{a} = \frac{1}{4}$. $\sin(2a) = \frac{1}{2}$ for $0 \le a \le 2\pi$. $a = \left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}$. Sum equals $3\pi$.

24. $\lim_{n\to\infty}\left|\frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x\ln n}{\ln(n+1)}\right| = |x| < 1$. $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges by the alternating series test and $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by the comparison test with $\sum_{n=2}^{\infty}\frac{1}{n}$. $[-1, 1)$. **C**

25. The integral is improper. If $\int_0^2 \frac{1}{(x-1)^3}\,dx$ converges, then it is equal to

$\int_0^1 \frac{1}{(x-1)^3}\,dx + \int_1^2 \frac{1}{(x-1)^3}\,dx$. However, both integrals diverge. Therefore $\int_0^2 \frac{1}{(x-1)^3}\,dx$ diverges. **D**

26. The velocity is $\mathbf{r}'(t)=\cos t\,\mathbf{i}+\left(30t^4+12t^2+4\right)\mathbf{j}$. $\mathbf{r}'(0)=\mathbf{i}+4\mathbf{j}$. The speed is therefore $\sqrt{17}$. **D**

27. Let $u=\sqrt{x^2+16}$. Squaring both sides gives $u^2=x^2+16$. As a result, $u\,du=x\,dx$ and $x^2=u^2-16$. Substituting this into the original integral yields $\int x^5\sqrt{x^2+16}\,dx=\int\left(u^2-16\right)^2u^2\,du=\int\left(u^6-32u^4+256u^2\right)du$. Integrating and subbing back in gives $\dfrac{\left(x^2+16\right)^{\frac{7}{2}}}{7}-\dfrac{32\left(x^2+16\right)^{\frac{5}{2}}}{5}+\dfrac{256\left(x^2+16\right)^{\frac{3}{2}}}{3}+C$. Factoring out $\left(x^2+16\right)^{\frac{3}{2}}$ and simplifying gives $\dfrac{\left(x^2+16\right)^{\frac{3}{2}}\left(15x^4-192x^2+2048\right)}{105}+C$. **D**

28. First note that the integral is improper.

$$\int_0^1\frac{8\operatorname{Cos}^{-1}x}{\sqrt{1-x^2}}\,dx=\lim_{b\to1}\int_0^b\frac{8\operatorname{Cos}^{-1}x}{\sqrt{1-x^2}}\,dx=\lim_{b\to1}\frac{8\left(\operatorname{Cos}^{-1}x\right)^2}{2}\Bigg]_0^b=4\lim_{b\to1}\left(\left(\operatorname{Cos}^{-1}b\right)^2-\left(\frac{\pi}{2}\right)^2\right)=$$

$4\left(0-\dfrac{\pi^2}{4}\right)=\pi^2$. $\log_\pi\pi^2=2$. **C**

29. As $P$ approaches $L$, $\dfrac{dP}{dt}$ approaches 0. Therefore $L$ is the carrying capacity. As a result, if $P>L$, the population will decrease. When $P$ is small, $\dfrac{dP}{dt}\approx kP$, and is approximately exponential. The largest rate of increase occurs at the inflection point, which is at $P(t)=\dfrac{L}{2}$. I, III, and IV are true. **C**

30. Partial fractions can be used to determine $\dfrac{1}{x^2-a^2}=\dfrac{1}{2a}\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right)$.

$$\int_{a+1}^{\infty}\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\int_{a+1}^{\infty}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx=\frac{1}{2a}\lim_{b\to\infty}\ln\left|\frac{x-a}{x+a}\right|\Bigg]_{a+1}^{b}$$

$$=\frac{1}{2a}\lim_{b\to\infty}\left(\ln\left|\frac{b-a}{b+a}\right|-\ln\left|\frac{1}{2a+1}\right|\right)=\frac{1}{2a}\left(0-\ln\left|\frac{1}{2a+1}\right|\right)=\frac{\ln(2a+1)}{2a}.$$ **C**