ALPHA ANALYTIC GEOMETRY
FAMAT State convention 2003

1) **A** Solving for when $x - \log_x 4$ equals zero, $x = 0, x = 2$, and as x approaches infinity, y approaches 0.

2) **C** The inverse of the function, *f*, is shown in answer b). This is the rotation of *f* about the line $y = x$. The opposite of the inverse, therefore, would be shown by rotating this graph about the *x* axis.

3) **E** Using proportionality, the height of the whole cone is $\frac{3}{2}$. The volume of the frustum is therefore $V_{large} - V_{small}$ for the two cones, or:

$$\Delta V = \frac{1}{3}(bh_{large} - bh_{small})$$

$$= \frac{1}{3}(36\pi \cdot \frac{9}{2} - 4\pi \cdot \frac{3}{2}) = 52\pi$$

4) **B** These two equations represent an ellipse with minor and major axes 3 and 5, respectively. The minimum value of *r*, therefore, is 3, since that is the smallest radius that the ellipse has.

5) **C** This point is the weighted Centroid of the 5 points given, or:

$$\bar{x} = \frac{10 \cdot (0+2+3+2) + 30 \cdot 0}{70(\textit{totalweight})} = 1$$

$$\bar{y} = \frac{10 \cdot (0+0+2+4) + 30 \cdot 3}{70} = \frac{15}{7}$$

6) **D**

$$\frac{f^2 - g^2}{f^2 + g^2} = \cos^2 x - \sin^2 x = \cos(2x)$$

and $\cos(2x)$ has roots at the points $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

7) **D** The n^{th} pyramidal number is found by summing the first n triangular numbers. Therefore, $P_{100} - P_{99} = T_{100}$, where T_{100} is the 100^{th} triangular number, and

$$T_n = \frac{n(n+1)}{2}$$

8) **C**

$4\sin\theta\cos\theta = 2\sin(2\theta)$

For $r = a\sin(b\theta)$ where b is even, the number of petals, $p = 2b$.

9) **A** The side length of the inside square, s', is in the relation $\sin\theta = \frac{x}{s'}$, therefore $s' = x\csc\theta$ and

$$A = (s')^2 = (x\csc\theta)^2$$

10) **B** There are many parametric linear representations of this line, and b) is one of them. a) and d) are not linear relationships, and c) is the parametric representation for the line $y = x - 1$, which does not contain those points.

11) **C** If we call the shaded area of the first 'Triforce' A, then the total shaded area of the 'Triforce' in Fig. 2 is $\frac{3}{4}A$, the shaded area of Fig. 3 is $\frac{3}{4}\left(\frac{3}{4}A\right)$, and so forth. Eventually, as we look closer and closer, the area, $\left(\frac{3}{4}\right)^N A$ as $N \to \infty$ equals 0.

12) **B** In exponential decay, $A = Pe^{rt}$. Solving for r,

$50\% = 100\% e^{42r}$ and $r = \frac{-\ln 2}{42}$.

So, $5\% = 100\% e^{rt}$, and $t \approx 181.5$

But since we are concerned with how many *more* days since Oct 12, the answer is $181.5 - 42 = 139.5$

13) **C** Solving the system

$2 = 4A + 2B + C$

$8 = 9A + 3B + C$,

$18 = 16A + 4B + C$

we get $A = 2, B = -4, C = 2$, yielding the parabola $y = 2x^2 - 4x + 2$. Plugging in $x = 5$ we get **32**

14) **A** The distance from the center to the focus of any hyperbola is $\sqrt{a^2 + b^2}$. Thus, the distance between the two foci would be $2\sqrt{3^2 + 4^2} = 10$

15) **B** The latera recta of the ellipse are of length $\frac{2b^2}{a} = \frac{2 \cdot 9}{4} = \frac{9}{2}$. The distance from the center to the latus rectum of the ellipse is $\sqrt{a^2 - b^2} = \sqrt{7}$. Since we must use twice this distance for the rectangle, the area is $\frac{9}{2} \cdot 2\sqrt{7} = 9\sqrt{7}$.

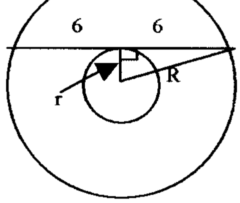
16) **B** This hyperbola has asymptotes of $x = 1$ and $y = 4$. Since the center of the hyperbola is always found at the intersection of the asymptotes, the center is $(1,4)$.

17) **B** see following page

18) **D** The equation can be factored into $(3y - 2x + 1)(1 - x - y) = 0$, which are two non-parallel (intersecting) lines.

19) **D** A parabola with vertex (h,k) and focus $(h+p,k)$ can be written as $(y-k)^2 = 4p(x-h)$. In this case, the vertex is halfway between the focus and directrix, i.e. $(1,2)$, and $p = 2$.

20) **D** $A_{annulus} = \pi(R^2 - r^2)$ and



So, $R^2 - r^2 = 36$

21) **E** $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, and $e^x + e^{-x} \neq 0$ so there are no vertical asymptotes. The horizontal asymptotes are found when $x \to \infty, y = 1$ and $x \to -\infty, y = -1$. So the asymptotes are at $y = \pm 1$

22) **B**

$16e^{\frac{4\pi i}{3}} = 16\cos(\frac{4\pi}{3}) + 16i\sin(\frac{4\pi}{3})$

and the roots are found by:

$\sqrt[4]{r} = \sqrt[4]{16}cis\left(\frac{1}{4}\cdot\frac{4\pi}{3} + k\frac{360}{4}\right)$ for $k = 0,1,2,3$. The roots are then $1+\sqrt{3}i, -\sqrt{3}+i, -1-\sqrt{3}i, \sqrt{3}-i$

23) **D** noticing the pattern:

$(1+i)^0 = 1$

$(1+i)^4 = -4$

$(1+i)^8 = 16......$

$(1+i)^{4n} = (2i)^{2n}$

so, $(1+i)^{20} = (2i)^{10} = -1024$

24) **E** for any two vectors, u, v, the relationship $\cos\theta = \dfrac{u \bullet v}{\|u\|\cdot\|v\|}$ holds for the angle, θ, between the vectors. Therefore, for our two vectors,

$$\cos\theta = \frac{2\cdot -1 + 4\cdot 4 + 4\cdot 0}{\sqrt{2^2 + 4^3 + 4^2}\cdot\sqrt{(-1)^2 + 4^2}}$$

25) **D** To find a perpendicular vector to two others, take the cross product, i.e.:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ -1 & 4 & 0 \end{vmatrix} = -16\mathbf{i} - 4\mathbf{j} + 11\mathbf{k}$$, but since the opposite of this vector is also perpendicular, both a) and c) are correct answers.

26) **C** For ellipses and hyperbolas, $e_{ellipse} = \dfrac{\sqrt{a^2 - b^2}}{a}$ and $e_{hyperbola} = \dfrac{\sqrt{a^2 + b^2}}{a}$. For the ellipse, $a = 4, b = 3$, but for the hyperbola, $a = 3, b = 4$, so the sum, $e_{ellipse} + e_{hyperbola} = \dfrac{3\sqrt{7} + 20}{12}$

27) **A** The directrices of a hyperbola run parallel to the conjugate axis, in our case $x = 0$. The distance from the center, (0,0) to the directrices is given by $D = \dfrac{a}{e}$. From the solution above, $e = \dfrac{5}{3}$, and $a = 3$, so the directrices are located at $x = \pm\dfrac{9}{5}$

28) **C** Using Green's theorem, the area of this triangle can be found by

$$A = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x & -2y & 1 \end{vmatrix}$$, or $2A = 3xy$.

29) **B** For the spherical coordinates (ρ,θ,ϕ),

$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$, so our coordinates

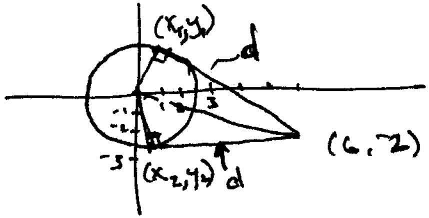
$z = \rho\cos\phi$

become $(\frac{15}{4}, \frac{5\sqrt{3}}{4}, \frac{5}{2})$ and $(2,0,0)$, yielding a distance of $\sqrt{14}$

30) **A** The area of every regular polygon is proportional to the square of its side length (i.e. $A = ks^2$), and a specific ***k*** exists for each *n*-gon. Therefore, for the pentagon,

$$\frac{A_L}{A_S} = \frac{k{s_L}^2}{k{s_S}^2} = \frac{{s_L}^2}{{s_S}^2} = \left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^2$$

17)



using Pythagorean theorem & distance formula,

$d = \sqrt{32} = 4\sqrt{2}$

The points (x_1, y_1) & (x_2, y_2) are located on the circle centered at $(6,-2)$ w/ $r = 4\sqrt{2}$, so $(6-x)^2 + (-2-y)^2 = 32$ and on $x^2+y^2=8$, so

$36 - 12x + x^2 + 4 + 4y + y^2 = 32$, $x^2+y^2=8$

$36 - 12x + 4 + 4y + 8 = 32$,

$y - 3x = -4$, $y = 3x - 4$

The points (x_1, y_1), (x_2, y_2) lie on the circle $x^2+y^2=8$ and the line $y = 3x-4$, so

$x^2 + (3x-4)^2 = 8 \Rightarrow x^2 + 9x^2 - 24x + 16 = 8$

$5x^2 - 12x + 4 = 0$

$(x-2)(5x-2) = 0$, $x = 2, \frac{2}{5}$

so, the points $(2,2)$ and $\left(\frac{2}{5}, \frac{-16}{5}\right)$ satisfy our requirements, and the lines

$y = 4 - x$ and $28y = 6x - 92$ connect those 2 points w/ $(6,-2)$