Theta Functions
FAMAT State Convention 2001

For all questions, answer E. "NOTA" means none of the above answers is correct.

1. Let $f(x) = 2x^2 + 1$ and $g(x) = \sqrt{\frac{x-1}{2}}$. Find $(f \circ g \circ g \circ f)(2001)$. (Note that $\circ$ denotes function composition).

A. $10\sqrt{10}$ B. 2000 C. 2001 D. 8008003 E. NOTA

2. Let $h(x) = x^3 + x$. Find $h^{-1}(10)$. (Note that $^{-1}$ denotes function inverse).

A. 1010 B. 2 C. $\frac{1}{1010}$ D. -1010 E. NOTA

3. A function $f(x)$ is called an *involution* if $f(x) = f^{-1}(x)$. How many of the following functions are involutions?

I. $f(x) = x$ II. $f(x) = x^2$ III. $f(x) = \sqrt{x}$ IV. $f(x) = \frac{1}{x}$ V. $f(x) = 2 - x$

A. 1 B. 2 C. 3 D. 4 E. NOTA

4. Let $h(x)$ be an even function with $h(xy) = h(x)h(y)$. What is $h(-1)$?

A. 1 B. -1 C. 0 D. Need more information E. NOTA

5. What are the domain restrictions on $f(x) = \sqrt{\frac{x-1}{x+1}}$, so that $f(x)$ is real-valued?

A. $-1 < x < 1$ B. $x \leq -1$ or $x > 1$
C. $-1 < x \leq 1$ D. $x < -1$ or $x \geq 1$ E. NOTA

6. A function $g(x)$ is called *idempotent* if $(g \circ g)(x) = g(x)$ for all x in its domain. Which of the following functions is **not** idempotent? (Note $\lfloor x \rfloor$ denotes the floor function of x and $\gcd(x, y)$ denotes the greatest common divisor)

A. $f(x) = -x$ B. $g(x) = \lfloor x \rfloor$ C. $h(x) = |x|$ D. $y(x) = \gcd(x, 24)$ E. NOTA

7. Let $h(x) = \log(x^2 + 2x + 2)$. Which of the following is a root of $h(x)$?

A. 1 B. $\frac{3}{2}$ C. 2 D. $\frac{5}{2}$ E. NOTA

8. Let $r(x)$ be a polynomial of order 4. If $r(x) \geq 0$ for all x, $r(1) = r(-1) = 0$, and $r(0) = 1$, then find r(2).

A. 0 B. 1 C. 3 D. 9 E. NOTA

9. Define $q(x) = (\log x)(x^2 - 4)$. What is the sum of the roots of $q(x)$?

A. 0 B. 1 C. 3 D. 5 E. NOTA

10. Let the point $(c, f(c))$ be an intersection point of $f(x)$ and its inverse, $f^{-1}(x)$. Find $\frac{f(c)}{c}$.

A. -1 B. 0 C. 1 D. Cannot be determined E. NOTA

11. Let a real-valued function $f(x)$ be defined over the integers. Suppose $f(x+y) = f(x)f(y)$ for all integers x and y. If $f(4) = 4$ then what is $f(11)$?

A. 11 B. 16 C. 32 D. $32\sqrt{2}$ E. NOTA

12. Which of the following functions are even?

I. x II. $\frac{1}{x^2}$ III. $\log|x|$ IV. 10^x

A. I and IV B. II only C. II and III D. IV only E. NOTA

13. Define $f(x) = x^2$ and $g(x) = -x^3$ and $h(x) = (f \circ g + g \circ f)(x)$. Which of the following are true?

I. $h(x)$ is an odd function II. $h(x)$ is an even function

A. I only B. II only C. both I and II D. neither I nor II E. NOTA

14. Given $v(x) = (x-2)^2(x+2)^2(x-1)(x+1)$, find the range of $v(x)$, with x real.

A. $v(x) \geq 0$ B. $v(x) \leq 0$ C. $-16 \leq v(x) \leq 0$ D. $v(x) \geq -16$ E. NOTA

15. Consider two concentric circles – one with radius x and the other with radius $x+2$. Define $f(x)$ as the area of the annulus between the two circles and $g(x)$ as the sum of the circumferences of the two circles. For $x > 0$, find $\frac{f(x)}{g(x)}$.

A. 1 B. 2 C. π D. Cannot be determined E. NOTA

16. A function is said to be *multiplicative* if for all x and y in its domain with $\gcd(x, y) = 1$, $f(xy) = f(x)f(y)$. Which of the following functions is **not** multiplicative?

A. $f(x) = x$, the identity function.
B. $\tau(x)$, Euler's totient function, equal to the number of positive integer divisors of x.
C. $\sigma(x)$, the sum of all the positive integer divisors of x.
D. $\omega(x)$, the number of distinct prime factors of x.
E. NOTA

17. Find the vertical asymptotes of $y = (g \circ f)(x)$ when $f(x) = \frac{x+1}{x+2}$ and $g(x) = \frac{1}{x(x-2)}$.

A. $x = 0$, $x = 2$ B. $x = -1$, $x = 1$
C. $x = 0$, $x = -2$ D. $x = -1$, $x = -3$ E. NOTA

18. Define $h(n) = \Pi_{j=1}^{n} 2^j$. Find $\log_2(h(8))$.

A. 6 B. 36 C. 55 D. 72 E. NOTA

19. Let $y = f(x)$ and $y = g(x)$ be the asymptotes of the hyperbola $x^2 - 4y^2 = 4$. Find $(f+g)(1)$.

A. -1 B. 1 C. 2 D. 4 E. NOTA

20. Suppose $p(x)$ is a fifth degree real-valued polynomial. If $1+i$ and $2-i$ are roots, how many real roots does $p(x)$ have?

A. 0 B. 1 C. 2 D. 3 E. NOTA

21. Let $f(x)$ be a linear function with a positive slope and $f(x) = 4f^{-1}(x) - 5$. What is $f(2)$?

A. -6 B. -1 C. $\frac{7}{3}$ D. 2 E. NOTA

For 22 and 23, use the following information: $f(x)$ and $g(x)$ are real-valued functions that satisfy

$$f(x-y) = f(x)f(y) + g(x)g(y) \qquad f(0) = 1$$
$$g(x-y) = g(x)f(y) - f(x)g(y) \qquad g(0) = 0$$

22. What is $(f(2))^2 + (g(2))^2$?

A. -1 B. 0 C. 1 D. Cannot be determined E. NOTA

23. Which of the following are true?

I. $f(x)$ is even II. $f(x)$ is odd
III. $g(x)$ is even IV. $g(x)$ is odd

A. I and IV B. II only C. II and III D. IV only E. NOTA

24. Suppose $h(x)$, which is not everywhere zero, has a well-defined inverse function $h^{-1}(x)$ that satisfies $(h^{-1} \circ h)(x) = x$ for all x in the domain of h. Which of the following **must always** be false?

A. $h(x)$ is even B. $h(x)$ is odd
C. $h(x)$ is one-to-one D. $h(0) = 1$ E. NOTA

25. Suppose $f(x)$ and $g(x)$ are invertible functions with $f(3) = 2$, $f(1) = 5$, $g(1) = 5$, and $g(3) = 1$. Find $(f \circ g^{-1} \circ f^{-1} \circ g)(1)$.

A. 5 B. 3 C. 2 D. 1 E. NOTA

26. Define $f(x) = \sqrt{x+2}$ and $g(x) = x^2$. If $(f \circ g \circ h)(x) = \sqrt{x^2+2x+3}$, then which of the following are possible definitions of $h(x)$?

A. $x+1$ B. $x-1$ C. x D. 1 E. NOTA

27. For $x > 0$ and $x \neq 1$, if $\log(x + g(x)) = \log(x) + \log(g(x))$, find $g(x)$.

A. x B. $\frac{x-1}{x}$ C. $\frac{1}{x}$ D. $\frac{x}{x-1}$ E. NOTA

28. Let $f(x) = x^5 - 3x^4 + 4x^3 - 4x^2 - 5x + 1$. How many negative roots does $f(x)$ have?

A. 0 B. 1 C. 2 D. 3 E. NOTA

29. If $y(t) = 2\log(t)$ and $x(t) = 3\log(t)$, what is y as a function of x?

A. $y = x^{\frac{2}{3}}$ B. $y = \frac{3}{2}x$ C. $y = 2x^3$ D. $y = \frac{2}{3}x$ E. NOTA

30. The Möbius function is defined over integers as follows:

$$\mu(x) = \begin{cases} -1 & \text{if } x \text{ is the product of an odd number of distinct primes} \\ 1 & \text{if } x \text{ is the product of an even number of distinct primes} \\ 0 & \text{if } x \text{ contains any multiple prime factors} \end{cases}$$

Determine $\mu(26)$.

A. -1 B. 0 C. 1 D. 26 E. NOTA