Calculus Applications SOLUTIONS
FAMAT State Convention 2001

1. $y = mx + b$; $5 = m \times 0 + b \Rightarrow b = 5$;
$2 = 4m + 5 \Rightarrow m = -3/4$ ANSWER B

2. Only choice D is possible by the intermediate value theorem. ANSWER D

3. $F = \int_0^{\pi} x\,dx = x^2/2\Big|_0^{\pi} = \pi^2/2$,

$G = \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 2 \;\therefore\; FG = \pi^2$

ANSWER D

4. $v(t) = 200 - 1.55t \Rightarrow h(t) = 200t - .775t^2 + h_0$; at $t = 0$, $h = 0 \Rightarrow h_0 = 0$; find t when $h(t)$ is 0 again, or $t(200 - .775t) = 0 \Rightarrow 200 - .775t = 0 \Rightarrow t \approx 258$ s
ANSWER A

5. $r(x) = xp(x) = x[3 - (x/40)]^2$;
$\frac{dr}{dx} = 9 - \frac{3}{10}x + \frac{3}{1600}x^2$; $\frac{dr}{dx} = 0$ @ $x = 40$ and 120;
only 40 is possible; $p(40) = \$4.00$ ANSWER E

6. $f(x)$ and $f'(x)$ must be continuous at 0 and π; all choices meet the first criteria, but only choice B meets the second criteria. ANSWER B

7. $2x^2 + 3y^2 = 5 \Rightarrow dy/dx = -2x/3y$;
$y^2 = x^3 \Rightarrow dy/dx = 3x^2/2y$; to be orthogonal, the product of the slopes must equal -1, or $-x^3/y^2 = -1$; all four choices fit this criteria, however only choices I and II also lie on the two curves. ANSWER A

8. $S = 4\pi r^2$; $dS = 8\pi r dr$; $dS = 8\pi(6371.4)(.16)$ or $dS = 25621$ km^2, or Maryland ANSWER B

9. This is the least squares problem. Basically, we define a function, J, equal to:

$$J = (y_1 - mx_1 - 1)^2 + (y_2 - mx_2 - 1)^2 + (y_3 - mx_3 - 1)^2 + (y_4 - mx_4 - 1)^2$$

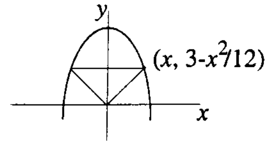
replacing the x- and y-values into the above gives:

$$J = (-1.5 + 2m)^2 + (0 - 0m)^2 + (1 - m)^2 + (2 - 3m)^2$$

or $J = 14m^2 - 20m + 7.25$. The minimum value of J will be when the derivative equals 0.

$dJ/dm = 0 = 28m - 20 \Rightarrow m = 5/7$ ANSWER A

10. The area of the triangle is given by the formula:



$A = 0.5(2x)(3-x^2/12)$
$dA/dx = 3-x^2/4 = 0$
or $x = 2\sqrt{3}$
Thus, $A = 4\sqrt{3}$

ANSWER C

11. A circular sector is defined as two radii of a circl and the arc connecting them. The perimeter of such a sector would be defined as: $p = 2r + r\theta$, where r is the radius of the circle and θ is the angle made by the two radii. The sector's area is given by $A = (\theta/2\pi)\pi r$ or $A = \theta r^2/2$. Since $\theta = (100/r) - 2$, $A = 50r - r^2$
$dA/dr = 50 - 2r = 0 \Rightarrow r = 25 \Rightarrow A = 625$.
ANSWER D

12. Given $T = 2\pi\sqrt{L/g}$ and $dL/d\theta = kL$, find $dT/d\theta$
From the first equation, we have $dT = \left(2\pi/2\sqrt{Lg}\right)dL$.
Dividing both sides by $d\theta$ and substituting the seconc equation gives $dT/d\theta = \pi k\sqrt{L/g} = kT/2$.
ANSWER C

13. Newton's method states that given an initial guess of a root of $f(x)$, x_0, a better value for the root is given by: $x_1 = x_0 - f(x_0)/f'(x_0)$. For this problem:

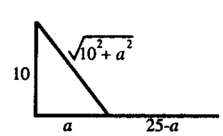
$$x_1 = 1.25 - \frac{1.25^2 - 1 - 0.5\sin(1.25^{\text{rad}})}{2(1.25) - 0.5\cos(1.25^{\text{rad}})} = 1.21243$$

ANSWER B

14. $y = uv$, so $dy/dt = u\,dv/dt + v\,du/dt$. Now, $du/dt = .04u$ and $dv/dt = .05v$. Therefore, $dy/dt = .05uv + .04uv = .09uv = d(uv)/dt$; the rate of growth of the total production is 9%. ANSWER C

15. $dv/dt = k\sqrt{v} \Rightarrow \int dv/\sqrt{v} = \int k\,dt$
$\Rightarrow 2\sqrt{v} = kt + C$. At $t = 0$, $v = 16$ and at $t = 4$, $v = 0$.
Thus $C = 8$ and $k = -2$. Thus, $v = 16 - 8t + t^2$.
Integrating wrt time gets, $x = 16t - 4t^2 + t^3/3 + x_0$.
$x(t = 4) - x(t = 0) = 64/3$ ANSWER C

16.



$Cost = 50000\sqrt{100 + a^2} + 30000(25 - a)$
$dC/da = 50000a/\sqrt{100 + a^2} - 30000 = 0$ or a=7.5 km.
Thus 25-a=17.5 km ANSWER C

17. $dv/dt = -g \Rightarrow v = -gt + v_0$. $v_0 = 0$. Integrating gives $h = -g\left(t^2/2\right) + h_0$. $h_0 = 10$. The height of the diver is at 0 when the time equals $\sqrt{20/g} = 1.429$ s. At this time, the velocity v is -14 ms^{-1}. ANSWER D

18. $v = 6\sin 3t$. Integrating gives $s = -2\cos 3t + s_0$. By the initial conditions, $s_0 = 2$; from $t = 0$ to $\pi/3$, v is positive; from $t = \pi/3$ to $2\pi/3$, v is negative; and from $t = 2\pi/3$ to π, v is positive again. So the total distance travelled will be

$$|s(\pi/3) - s(0)| + |s(2\pi/3) - s(\pi/3)| + |s(\pi) - s(2\pi/3)|$$

or 4+4+4=12 ANSWER A

19. The average daily inventory is

$$\frac{1}{30-0}\int_0^{30}(1200 - 40x)dx = \frac{1}{30}\left(1200x - 20x^2\right]_0^{30} = 600$$

The average daily holding cose for the chocolate is the dollar cost of holding one case times the average daily inventory: $(\$0.05)(600) = \$30$ ANSWER B

20. $y = \sqrt{x} \Rightarrow dy = dx/\left(2\sqrt{x}\right) \Rightarrow dy = 1/\left(2\sqrt{64}\right)$
$\Rightarrow dy = 1/16$. $\therefore \sqrt{65} \approx 8 + (1/16) = 8.0625$
ANSWER C

21. Simpsons approximation for $\int_a^b f(x)dx$ is

$$S = \frac{b-a}{3n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

$$S = \frac{4}{3(6)}\left(0 + 4\sqrt{\frac{2}{3}} + 2\sqrt{\frac{4}{3}} + 4\sqrt{2} + 2\sqrt{\frac{8}{3}} + 4\sqrt{\frac{10}{3}} + 2\right)$$

or $S \approx 5.29$ ANSWER A

22. Work equals the total change in kinetic energy.

$KE_{final} - KE_{initial} = 0.5mv_{final}^2 - 0.5mv_{initial}^2$
$= 0.5(.06\text{ kg})\left(60\text{ ms}^{-1}\right)^2 = 108\text{ N-m}$ ANSWER B

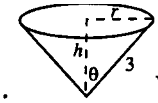
23. Use the u-substitution, $u = \sqrt[6]{x}$. Then the integral becomes

$$\int_1^2 \frac{6u^5du}{u^3 + u^2} = \int_1^2 \frac{6u^3du}{u+1} = \int_1^2\left(6u^2 - 6u + 6 - \frac{6}{u+1}\right)du$$

$= 2u^3 - 3u^2 + 6u - 6\ln|u+1|\Big|_1^2$
$= 11 + 6(\ln 2 - \ln 3) = 11 + 6\ln(2/3)$ ANSWER D

24. $\int_0^\infty \cosh x - \sinh x\, dx = \int_0^\infty \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}dx$

$\int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = 1$ ANSWER E

25. 

We know that $r^2 + h^2 = 9$.
We also know that $h = 3\cos\theta$ and $r = 3\sin\theta$. So,
$V = (1/3)\pi r^2h = 9\pi\sin^2\theta\cos\theta$.
$dV/d\theta = 9\pi\sin\theta\left(2\cos^2\theta - \sin^2\theta\right) = 0$
$\therefore \tan\theta = \sqrt{2}$ maximizes the volume. ANSWER D

26. A cross-section of this solid perpendicular to the x-axis is a square. If I integrate over dx the area of these squares, I will have the volume:

$$\int_{-3}^{3}\left(2\sqrt{9 - x^2}\right)^2dx = 4\left(9x - \frac{x^3}{3}\right]_{-3}^{3} = 144$$

ANSWER D

27. $dx/dt = kx \Rightarrow dx/x = kdt \Rightarrow \ln x = kt + C$
The initial conditions give
$C = \ln 2$ and $k = 0.1\ln 2$.
$\therefore \ln x = (0.1\ln 2)5 + \ln 2 = 1.5\ln 2$ and $x = 2\sqrt{2}$
ANSWER E

28. Newton's Law of Cooling: $dT/dt = k\left(T - T_s\right)$
$\Rightarrow T - T_s = \left(T_0 - T_s\right)e^{kt}$, where $T_0 = 99^\circ\text{C}$ and $T_s = 19^\circ\text{C}$. At 5 minutes, $-\ln 4 = 5k$, which solves for k. When the water goes to room temperature, $-\ln 80 = (-0.2\ln 4)t \Rightarrow t = 10 + 5\log_4 5$, which is $5 + 5\log_4 5$ minutes more. ANSWER A

29. $V = (4/3)\pi r^3 \Rightarrow dV = 4\pi r^2dr$
$\Rightarrow 1\text{ cm}^3 = 4\pi(10\text{ cm})^2dr \Rightarrow dr \approx 7.958\times10^{-4}$ cm.
The acid eats away at a rate of 0.1 mm/hr, or

$$\frac{dr}{dt} = \frac{.1\text{ mm}}{1\text{ hr}}\left|\frac{1\text{ cm}}{10\text{ mm}}\right|\frac{1\text{ hr}}{3600\text{ s}} = 2.778\times10^{-6}\text{ cm/s}$$

$dr/(dr/dt) \approx 286$ sec. ANSWER A

30. $\text{Rad}_{\text{insc. circle}} = \text{Area/semiperimeter}$
$= \sqrt{s(s-a)(s-b)(s-c)}/s$

$$= \frac{\sqrt{(3/2)(x+1)(1/2)(x+3)(1/2)(x+1)(1/2)(x-1)}}{(3/2)(x+1)}$$

$= \left(\sqrt{3}/6\right)\sqrt{(x+3)(x-1)}$
$\text{Area}_{\text{ins. circle}} = \pi\text{Rad}_{\text{ins. circle}}^2 = (\pi/12)(x+3)(x-1)$
$\text{A} = (\pi/12)\left(x^2 + 2x - 3\right)$ Since this is a quadratic equation, there will be no inflection points (the second derivative of A is a constant not equal to 0).
ANSWER D