Linear Algebra
FAMAT State Convention 2001

For all questions, answer E. "NOTA" means none of the above answers is correct.
For all questions A_{ij} or $A_{i,j}$ denotes the element in the i^{th} row and j^{th} column of A.

1) If $\vec{c} = 7i - 6j + 2k$ and $\vec{d} = 4i + 3j - 6k$, find $\vec{c} \cdot \vec{d}$

A) -2 B) 3 C) 28 D) 58 E) NOTA

2) If the shortest distance between the line $3x + 4y = 10$ and the point $(a, 6)$ is 7, find a.

A) 5 B) 6 C) 7 D) 8 E) NOTA

3) Which of the following lines contains the point $(-7, -4, 0)$ and lies in the direction $(4, 1, 11)$?

A) $l(t) = (-7t + 4)i + (-4t + 1)j + 11k$
B) $l(t) = (4t - 7)i + (t - 4)j + (11t)k$
C) $l(t) = (4t - 7)i + (t - 4)j + 11k$
D) $l(t) = (4t + 7)i + (t + 4)j + (11t)k$
E) NOTA

4) Find $[c]$, the greatest integer less than or equal to c, if the matrix $\begin{bmatrix} 7 & 9 & -1 \\ 3 & -2 & 6 \\ 6 & c & -5 \end{bmatrix}$ is singular.

A) 10 B) 11 C) 12 D) 13 E) NOTA

5) The rank of the matrix $\begin{bmatrix} 2 & 4 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & -1 & -1 \end{bmatrix}$ is:

A) 1 B) 2 C) 3 D) 4 E) NOTA

6) What is the sum of the entries in each row, column, and diagonal of a simple 10x10 magic square (one in which every integer in $[1, 10^2]$ appears exactly once)?

A) 500 B) 505 C) 510 D) 515 E) NOTA

7) Let $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$. Which of the following are eigenvectors of A^T ? :

A) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ B) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ C) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ E) NOTA

8) How many of the following statements are true?

- There exists a real matrix A such that $A^{17} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- If A is a matrix and is factored to LU, where L and U are A's upper-triangular and lower-triangular factors, respectively, $\det(A) = \det(LU)$
- If A is an nxn matrix and I is the nxn identity matrix, A and $A+I$ have the same eigenvectors.
- If a matrix A is diagonalizable, its eigenvalues are distinct.

A) 0 B) 1 C) 2 D) 3 E) NOTA

9) Which of the following matrices is **not** a rotation?

A) I, the identity matrix

B) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

C) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

D) $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

E) NOTA

10) Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

A) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ B) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ D) $\begin{bmatrix} -1 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ E) NOTA

11) Let q be the angle between $\vec{p} = (3,4,5)$ and $\vec{q} = (0,-7,1)$. If $|\cos(2q)|$ is expressed as $\frac{P}{Q}$, where $\gcd(P, Q) = 1$, find $\sqrt{Q-P}$.

A) 22 B) 23 C) 24 D) 25 E) NOTA

12) If A is any nxn matrix, l is an unknown scalar value, I is the identity matrix, and $q(l) = \det(A - lI)$, then $q(l)$ is the ___ of A.

A) Eigen polynomial B) Cayley-Hamilton polynomial
C) Determinant polynomial D) Characteristic polynomial E) NOTA

13) If $A=\begin{bmatrix}1&2&-1&0\\0&1&-1&0\\0&-1&0&1\\3&4&5&1\end{bmatrix}$ and $B=\begin{bmatrix}w&3&2&1\\1&x&2&1\\7&-1&y&-1\\6&0&1&z\end{bmatrix}$, and $\det(AB)=63$, find $\det(B)$.

A) -9 B) -7 C) 9 D) Not enough information E) NOTA

14) If $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, what is the dimension of the left-nullspace of A?

A) 0 B) 1 C) 2 D) 3 E) NOTA

15) Find the hundredth's digit of the area of the triangle defined by the following points: $(1,2,3)$, $(4,7,8)$, and $(3,2,6)$.

A) 2 B) 4 C) 6 D) 8 E) NOTA

16) Suppose T is an nxn matrix with eigenvalues $1,2,3,\ldots,n$ for odd n.

If $\sum_{i=1}^{\frac{n-1}{2}}\left(T_{i,i}+T_{n+1-i,n+1-i}\right)=\frac{n^2}{2}$, find $T_{\frac{n+1}{2},\frac{n+1}{2}}$.

A) $\frac{n-1}{2}$ B) $\frac{n}{2}$ C) $\frac{n+1}{2}$ D) $n+1$ E) NOTA

17) Find a least-squares solution $\vec{x}$ to $A\vec{x}=\vec{b}$, where $A=\begin{bmatrix}1&0\\1&2\\1&1\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

A) $\begin{bmatrix}1\\-1/3\end{bmatrix}$ B) $\begin{bmatrix}2/3\\0\end{bmatrix}$ C) $\begin{bmatrix}1/3\\1/3\end{bmatrix}$ D) No least-squares solution E) NOTA

18) Find the sum of the elements in e^K if $K=\begin{bmatrix}0&1\\0&0\end{bmatrix}$.

A) 1 B) 3 C) 5 D) $e+3$ E) NOTA

19) Let $q(x,\ y)=13x^2+24xy+13y^2$. If $q(x,\ y)=\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix}$, find $A_{12}+A_{21}$.

A) 13 B) 24 C) 26 D) 48 E) NOTA

For questions 20-21, let $M=\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

20) Which of the following matrices is similar to *M*?

A) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ B) $\begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}$ C) $\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$ D) $\begin{bmatrix} 4 & 0 \\ 0 & 1/2 \end{bmatrix}$ E) NOTA

21) Find $N_{11}-N_{12}$ if $N=M^{10}$.

A) -1 B) 0 C) 1 D) $2^{10}-1$ E) NOTA

22) Find the projection of $\vec{v}=(3,-2,5)$ onto $\vec{u}=(3,6,2)$.

A) $\frac{1}{9}\vec{u}$ B) $\frac{7}{37}\vec{u}$ C) $\frac{1}{7}\vec{v}$ D) $\frac{7}{39}\vec{v}$ E) NOTA

23) If $Z=\begin{bmatrix} 0 & 1 & 3 & 4 & 3 \\ 0 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 3 & 4 \\ 1 & 1 & 4 & 5 & 5 \end{bmatrix}$, Z_{re} is the row echelon form of *Z* with all pivots$=1$, and Z_{rre} is the row-reduced echelon form of *Z*, find the number of nonzero elements in $Z_{re}-Z_{rre}$.

A) 1 B) 2 C) 3 D) 4 E) NOTA

24) Suppose $\vec{v}=\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$ and $T(\vec{v})=\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$, where $f(x)=c_0+c_1x+c_2x^2+c_3x^3$.

If A is the $4x4$ matrix such that $T(\vec{v})=A\vec{v}$, find the trace of A.

A) 1 B) 4 C) 33 D) 40 E) NOTA

25) Which of the following statements is/are true?

I) If A is a square matrix whose nullspace consists only of the zero vector, A is invertible.

II) If $A = \begin{bmatrix} .1 & .2 \\ .3 & .4 \end{bmatrix}$, as n becomes large, A^n approaches the zero matrix.

III) Every real orthogonal matrix is also unitary

A) II only B) I and II only C) II and III only D) I, II, and III are all true E) NOTA

26) Let $D = \begin{bmatrix} 3 & 4+3i \\ 1-6i & -2 \end{bmatrix}$ and D^H be its Hermitian (conjugate transpose) matrix. If the product of the elements in D^H is expressed as $a+bi$, find $b-a$.

A) –268 B) -6 C) 6 D) 268 E) NOTA

27) Let A_i and B_i be $2x2$ matrices, and I be the identity matrix. If $B_i^{-1}A_i^{-1} = I$ for odd i and $B_i^{-1}A_i^{-1} = 2I$ for even i, find $\log_2\left(\prod_{i=0}^{2001}(A_iB_i)^{-1}\right)_{11}$.

A) 1000 B) 1001 C) 2000 D) 2001 E) NOTA

28) Which of the following forms an orthonormal basis spanning the same subspace of $\mathbf{R}^3$ as the basis: $\{(1,2,2),(1,3,1)\}$?

A) $\left\{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{2}}(1,0,-1)\right\}$

B) $\{(1,0,0),(0,1,0)\}$

C) $\left\{\frac{1}{3}(1,2,2), \frac{1}{\sqrt{2}}(0,1,-1)\right\}$

D) $\left\{\sqrt{3}\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \sqrt{2}\left(\frac{1}{\sqrt{12}},\frac{1}{\sqrt{12}},-\frac{1}{\sqrt{3}}\right)\right\}$

E) NOTA

29) 3 out of every 4 trucks on I-75 follow a car, and 1 out of every 5 cars on I-75 follows a truck. If all vehicles on I-75 are either cars or trucks, what is the proportion of trucks on I-75?

A) $4/19$ B) $3/11$ C) $2/7$ D) $1/3$ E) NOTA

30) Let the i, j, and k axes be oriented on the page horizontally, vertically, and through the page, respectively, with the positive axes pointing to the right, upward, and out of the page, respectively. If $\vec{v_1} = (9,4,0)$ and $\vec{v_2} = (7,3,0)$, describe $\vec{v_1} \times \vec{v_2}$.

A) 1 unit out of the page B) 1 unit into the page
C) 75 units upward to the right D) 75 units downward to the left E) NOTA