# Linear Algebra

FAMAT State Convention 2000

For all questions, answer "E) NOTA" stands for "none of the above."

1. Solve the system of equations:

$$2x+y=14$$
$$x+5y+z=16$$
$$3x+2y-2z=3$$

A) (2,-5,1) B) (5,4,0) C) (7,0,9) D) (8,-2,3) E) NOTA

2. Which of the following matrices is in reduced row echelon form?

A) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 5 \\ 1 & 0 \end{bmatrix}$ C) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ D) $\begin{bmatrix} 1 & -7 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ E) NOTA

3. Find all vectors in $\Re^3$ that are perpendicular to $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (s,t are arbitrary constants).

A) $\left\{ \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \right\}$ B) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ C) $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{-1}{3} \end{bmatrix} \right\}$ D) $s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ E) NOTA

4. Given a system with p linear equations, q unknowns, and a coefficient matrix A, if rank(A) = q ≤ p, then which of the following statements must be true?

I. A is a p by q matrix.
II. The system is consistent.
III. The system has infinitely many solutions.
IV. The system has, at most, one solution.

A) I, II, IV B) I, IV C) II, III D) I, II, III E) NOTA

5. Given the linear transformation $T(\vec{x}) = A\vec{x}$ from $\Re^2$ to $\Re^2$ that rotates any vector $\vec{x}$ clockwise through an angle of 30°, find the transformation matrix A.

A) $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ B) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ C) $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

D) $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ E) NOTA

6. Let $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 6 \\ 3 \\ \frac{5}{2} \\ 0 \end{bmatrix}$, and L be the line in $\Re^5$ consisting of all scalar multiples of $\vec{v}$. What is the reflection of $\vec{w}$ about the line L?

A) $\begin{bmatrix} 1 \\ -6 \\ 1 \\ \frac{-3}{2} \\ 2 \end{bmatrix}$ B) $\begin{bmatrix} 4 \\ 0 \\ \frac{16}{9} \\ \frac{4}{25} \\ 4 \end{bmatrix}$ C) $\begin{bmatrix} \frac{27}{25} \\ -6 \\ \frac{-71}{25} \\ \frac{-123}{50} \\ \frac{2}{25} \end{bmatrix}$ D) $\begin{bmatrix} \frac{-1}{9} \\ \frac{24}{25} \\ 1 \\ \frac{-42}{25} \\ \frac{27}{50} \end{bmatrix}$ E) NOTA

7. If $A = \begin{bmatrix} 2 & -4 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $B^{-1} = \frac{1}{25}\begin{bmatrix} 1 & -8 \\ 12 & 4 \end{bmatrix}$, what is $(AB)^{-1}$?

A) $\begin{bmatrix} \frac{-97}{150} & \frac{-4}{25} \\ \frac{-1}{3} & 0 \end{bmatrix}$ B) $\begin{bmatrix} \frac{-46}{25} & \frac{-32}{25} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$ C) $\begin{bmatrix} 0 & 3 \\ \frac{251}{4} & \frac{97}{8} \end{bmatrix}$ D) $\begin{bmatrix} \frac{1}{10} & \frac{4}{25} \\ \frac{-2}{15} & \frac{-56}{75} \end{bmatrix}$ E) NOTA

8. Given the linear transformation $T(\vec{x}) = A\vec{x}$ from $\Re^3$ to $\Re^2$ where $A = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 5 & -4 \end{bmatrix}$, which set of vectors spans the image of T?

A) $\left\{\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}\right\}$ B) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ C) $\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \end{bmatrix}\right\}$ D) $\left\{\begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}\right\}$ E) NOTA

9. What is the kernel of the linear transformation with transformation matrix $A=\begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 3\end{bmatrix}$?

A) $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$ B) $\{\vec{0}\}$ C) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ D) $\{\ \}$ E) NOTA

10. Which of the following is a subspace of $\Re^3$?

A) $\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}: x+y\geq z\right\}$ B) $\left\{\begin{bmatrix}x\\y\end{bmatrix}: xy\leq 0\right\}$ C) $\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}: x+y+z=4\right\}$

D) $\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}: x>y>z\right\}$ E) NOTA

11. Given the vectors $\begin{bmatrix}-1\\0\\2\end{bmatrix},\begin{bmatrix}3\\a\\4\end{bmatrix},\begin{bmatrix}b\\2\\0\end{bmatrix}$, which of the following values of a and b would make the vectors linearly independent?

A) a=10, b=1 B) a=3, b=2 C) a=-2, b=-4 D) a=4, b=5 E) NOTA

12. What is the acute angle between the vectors $\begin{bmatrix}-1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}4\\5\\3\end{bmatrix}$?

A) 56.2° B) 33.9° C) 80.1° D) 67.7° E) NOTA

13. What is the trace of the following matrix?

$$\begin{bmatrix} 5 & 2 & 4 & 0 & 1 \\ 4 & 0 & -1 & 2 & 5 \\ 3 & -6 & 3 & -6 & 3 \\ 1 & 10 & 4 & 11 & 5 \\ 0 & 1 & 1 & 4 & 9 \end{bmatrix}$$

A) 4 B) 5 C) 16 D) 23 E) NOTA

14. What is the rank of the following matrix?

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ -2 & 1 & 5 & 1 \\ 3 & 2 & 4 & 9 \\ -1 & 5 & 8 & 1 \end{bmatrix}$$

A) 3 B) 4 C) 7 D) 16 E) NOTA

15. Given $T(\vec{x}) = A\vec{x}$ from $\Re^n$ to $\Re^n$ is an orthogonal linear transformation, which of the following must be true concerning an arbitrary constant $k$ and vectors $\vec{v}, \vec{w}$ in $\Re^n$?

I. $\|T(\vec{v})\| = \|\vec{v}\|$

II. $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

III. $\|T(\vec{v})+T(\vec{w})\|^2 = \|T(\vec{v})\|^2 + \|T(\vec{w})\|^2$

IV. $kT(\vec{w}) = T(k\vec{w})$

A) I, IV B) I, II, IV C) I, III D) II, III, IV E) NOTA

16. The determinant of matrix A is 6 and the determinant of matrix B is 4. Evaluate:

$$\sqrt{\frac{|\,B^{-1}A^T\,|}{|\,(AB^T)^{-1}\,|}}$$

A) 1 B) 4 C) 6 D) 24 E) NOTA

17. What is the classical adjoint of $\begin{bmatrix} 3 & 6 & 1 \\ 0 & 2 & 5 \\ -1 & 4 & -1 \end{bmatrix}$?

A) $\begin{bmatrix} -22 & 10 & 28 \\ -5 & -2 & -15 \\ 2 & -18 & 6 \end{bmatrix}$ B) $\begin{bmatrix} 3 & 0 & -1 \\ 6 & 2 & 4 \\ 1 & 5 & -1 \end{bmatrix}$ C) $\begin{bmatrix} -282 & 0 & 94 \\ -564 & 188 & -376 \\ -94 & -470 & 94 \end{bmatrix}$

D) $\begin{bmatrix} 11 & -5 & -14 \\ \frac{5}{2} & 1 & \frac{15}{2} \\ -1 & 9 & -3 \end{bmatrix}$ E) NOTA

18. What is the volume of the parallelepiped defined by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ -9 \end{bmatrix}$?

A) 9 B) 11 C) 27 D) 33 E) NOTA

19. What is the expansion factor of the linear transformation $T(\vec{x}) = \begin{bmatrix} 6 & 4 \\ -1 & 3 \end{bmatrix}\vec{x}$ from $\Re^2$ to $\Re^2$?

A) 2 B) 3 C) 4 D) 5 E) NOTA

20. What is the numerator of the fraction used to solve for $y$ with Cramer's method in the following system when the variables are arranged in alphabetical order?

$$w+x+3z=-1$$
$$-5w+2x-y+z=4$$
$$3x+4w+5y-2z=10$$
$$3w+2y-z=-11$$

A. $\begin{vmatrix} 1 & 1 & 3 & -1 \\ -5 & 2 & -1 & 4 \\ 3 & 4 & 5 & 10 \\ 3 & 2 & -1 & -11 \end{vmatrix}$ B. $\begin{vmatrix} 1 & 1 & 0 & 3 \\ -5 & 2 & -1 & 1 \\ 4 & 3 & 5 & -2 \\ 3 & 0 & 2 & -1 \end{vmatrix}$

C. $\begin{vmatrix} 1 & 1 & -1 & 3 \\ -5 & 2 & 4 & 1 \\ 4 & 3 & 10 & -2 \\ 3 & 0 & -11 & -1 \end{vmatrix}$ D. $\begin{vmatrix} 1 & -5 & 3 & 3 \\ 1 & 2 & 4 & 0 \\ 3 & -1 & 5 & 2 \\ -1 & 1 & -2 & -1 \end{vmatrix}$ E. NOTA

21. What are the eigenvalues of the matrix $\begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$?

A) 3,-2 B) 5,-4 C) -6,14 D) -1,1 E) NOTA

22. What is the sum of the squares of the eigenvalues of a square matrix A if $|A| = 5$ and $tr(A) = -4$?

A) 13 B) 9 C) 1 D) 6 E) NOTA

23. What is the algebraic multiplicity of the eigenvalues of the matrix $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \end{bmatrix}$?

A) 3,1 B) 2,2 C. 1,1,2 D. 0,1,3 E. NOTA

24. What are the eigenvectors of $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

A) $\vec{0}$ B) $\text{span}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ C) $\text{span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ D) $\text{span}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ E) NOTA

25. Which set of vectors forms an eigenbasis for $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$?

A) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ B) $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ C) $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$ D) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ E) NOTA

26. Find the complex eigenvalues of $\begin{bmatrix} 4 & 5 \\ -4 & -4 \end{bmatrix}$.

A) $1 \pm 2i$ B) $-1 \pm 2i$ C) $\pm 2i$ D) $-1 \pm i$ E) NOTA

27. Given $\vec{x} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}$, find $[\vec{x}]_B$ where $B$ denotes the basis defined by $\left\{ \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 9 \end{bmatrix} \right\}$.

A) $\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$ B) $\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ C) $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} 6 \\ -7 \\ -4 \end{bmatrix}$ E) NOTA

28. Compute: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{11}$

A) $\begin{bmatrix} 11 & 0 & 11 \\ 0 & 11 & 0 \\ 11 & 0 & 11 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 22 & 0 & 22 \\ 0 & 22 & 0 \\ 22 & 0 & 22 \end{bmatrix}$

D) $\begin{bmatrix} 11 & 0 & 1 \\ 0 & 11 & 0 \\ 1 & 0 & 11 \end{bmatrix}$ E) NOTA

29. Matrix B is obtained by performing a row swap on matrix A. What is the relationship between the determinants of these two matrices?

A) $\det(A) = \det(B)$ B) $\det(A) = -\det(B)$ C) $\det(A) = \det(B)^{-1}$

D) $\det(A) = -\det(B^{-1})$ E) NOTA

30. Which method is used to find an orthonormal basis of a subspace of $\Re^n$?

A) Gauss-Jordan Elimination B) Gram-Schmidt Process C) Cramer's Method
D) Cauchy-Schwarz Rule E) NOTA